Stochastic Joint Replenishment Problems: Periodic Review Policies

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Abstract

Operations Managers of manufacturing systems, distribution systems, and supply chains address lot sizing and scheduling problems as part of their duties. These problems are concerned with decisions related to the size of orders and their schedule. In general, products share or compete for common resources and thus require coordination of their replenishment decisions whether replenishment involves manufacturing operations or not.

This research is concerned with joint replenishment problems (JRPs) which are part of multi-item lot sizing and scheduling problems in manufacturing and distribution systems in single echelon/stage systems.

The principal purpose of this research is to develop three new periodic review policies for stochastic joint replenishment problem. It also highlights the lack of research on joint replenishment problems with different demand classes (DSJRP). Therefore, periodic review policy is developed for this problem where the inventory system faces different demand classes that are deterministic demand and stochastic demand. Heuristic Algorithms have been developed to obtain (near) optimal parameters for the three policies as well as a heuristic algorithm has been developed for DSJRP. Numerical tests against literature benchmarks have been presented.

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Adel F Alrasheedi*)

To my parents

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Operations managers of manufacturing systems, distribution systems, and supply chains address lot sizing and scheduling problems as part of their duties. These problems are concerned with decisions related to the size of orders and their schedule. In general, products share or compete for common resources and thus require coordination of their replenishment decisions whether replenishment involves manufacturing operations or not.

Joint replenishment problems (JRPs) are concerned with coordinating replenishment decisions of several products in situations where the joint replenishment of a subset of products or all products achieves some sort of economies of scale by sharing, for example, fixed ordering costs, setup costs, or transportation costs so as to minimise the total costs. Typically, each time a replenishment order is placed, an ordering cost is incurred consisting of a major fixed cost, often referred to as the joint replenishment ordering cost, as well as a minor cost for each item on the order, often referred to as item-specific ordering costs, where replenishment could involve manufacturing or not. In many practical situations, it is more common to use joint replenishment of items than the independent replenishment of a single item. Substantial savings may be obtained when several items are ordered using the same truck (e.g., transportation costs) or the same machine. Therefore, an efficient joint replenishment policy is required. If the operations mangers effectively apply the concept of the JRPs in their manufacturing/distribution systems, they could join the replenishment order of the items to meet customers demand and significantly, to reduce substantial cost in the meanwhile. The main goal for using inventory replenishment policy include keeping stock avoiding both the problem of oversupply of inventory, the problem of stockouts, and minimise the total cost of replenishment.

In some industries (e.g., food industry, chemical industry), managers often have to deal with a variant of the JRP where products are manufactured in large batches (e.g., full milk, semi-skimmed milk, skimmed milk), then packaged into various types of containers (e.g., 50ml, 100ml, 200ml) commonly referred to as items (e.g., 50ml container of full milk, 50ml container of semi-skimmed milk, 100ml container of skimmed milk), and there is a primary setup cost for the manufacture of the product (e.g., full milk) and a secondary setup cost for each of the items to be packaged (e.g., 50ml container of full milk, 100ml container of full milk, 200ml container of full milk). Therefore, the joint replenishment problem is what products (e.g., full milk) should be manufactured and items (e.g., container of full milk, 50 ml) should be packaged in each manufacturing run in order to determine the minimum manufacturing setup costs and packaging setup costs. In distribution system, if several items are ordered from the same supplier, the joint ordering cost can be shared by ordering two or more items jointly. In sum, this variant involves two types of operations; namely, manufacturing and packaging, whereas the JRP involves only one type of operations such as manufacturing or distribution. A similar variant involves manufacturing and assembly operations.

A stochastic joint replenishment problem (SJRP) is a version of JRP when demand for each item is stochastic and stationary. The objective is to minimize the expected total costs per unit time where the expected total cost consists of three costs; expected ordering cost, expected inventory cost and expected shortage cost. Two main policies have been most commonly used in the literature for addressing the SJRP; namely, continuous review policies and periodic review policies. We have completed an extensive literature review on deterministic joint replenishment problems (JRPs) and stochastic joint replenishment problems (SJRPs) and their inventory control policies. This led to the development of three new policies for the stochastic joint replenishment problems. It also highlighted the lack of research on joint replenishment problems with different demand classes. This thesis focuses on stochastic joint replenishment problems (SJRPs).

The main contribution of this thesis to the research on the JRPs is to propose three periodic review policies for SJRPs. The first one which is referred to as (mF, s, S) policy. The policy is a periodic review policy for SJRPs. The principle of this policy is that the inventory for each item i is reviewed periodically every m_iF time units and if the inventory position is below their s, order up to level S, m is a positive integer. In addition, a heuristic Algorithm has been developed to search for near optimal parameters for an (mF, S) policy. The second one which is referred to as (F, Q, S) policy, where the inventory is reviewed periodically every F time units, and all items are replenished up to their levels S only if the aggregate demand during a replenishment order cycle reaches Q. The proposed policy bases the replenishment decisions only on the total demand that have accumulated for all items during replenishment order cycle. For this policy, we developed expressions for the operating characteristics of the inventory system and constructed the expected total cost function for Poisson demand process. Numerical results have been conducted to study the sensitivity of the policy to various system parameters and to evaluate the performance of the proposed policy over existing policies. The third policy is referred to as (F, Q, s, S) policy. This policy is to add a must-order point for each item in the (F, Q, S) policy. Moreover, a periodic review policy is developed for this problem where the inventory system faces different demand classes that are deterministic demand and stochastic demand. Heuristic Algorithms have been developed to search for near optimal parameters for the three policies and a heuristic algorithm has been developed for an $(mF, R+S)$ policy. Numerical tests against literature benchmarks have been presented.

The thesis is structured as follows. Chapter 2 presents an extensive literature review on deterministic and stochastic joint replenishment problems and the critical analysis of the literature. Chapter 3 is concerned with the periodic review policy for SJRPs, denoted (mF, s, S). It deals with two control variables where F is continuous and m is positive integer. Also, it deals with a new heuristic algorithm for the (mF, S) policy. Chapter 4 is concerned with the two periodic review policies for SJRPs; namely, (F, Q, S) policy and (F, Q, s, S) policy. It deals with two decision variables where F is continuous and Q is discrete. Chapter 5 deals with the joint replenishment problems with different demand classes. The periodic review policy $(mF, R + S)$ is proposed for the problem. Chapter 6 summarizes the results obtained, presents some conclusions, and makes some suggestions for future research.

Chapter 2

Literature Review and Classification

This chapter is concerned with joint replenishment problems (JRPs) which are part of multi-item lot sizing and scheduling problems in manufacturing and distribution systems in single echelon/stage systems. An extensive literature review on deterministic joint replenishment problems (JRPs) and stochastic joint replenishment problems (SJRPs) and their inventory control policies are presented with a particular focus on the stochastic joint replenishment problems (SJRPs).

The chapter is organized as follows. We provide a description of the JRPs, and a classification of the literature in section 2.1 and 2.2 respectively. Typical assumptions for the model, the model formulation and a critical analysis of the literature for the deterministic JRPs are discussed in Sections 2.3. Also, we review the stochastic JRPs in section 2.4.

2.1 Introduction

Joint replenishment problems (JRPs) are part of multi-item lot sizing and scheduling problems in manufacturing and distribution systems in single echelon/stage systems. The joint replenishment problem is concerned with the determination of the optimal replenishment for multi-item inventory system and stocking decisions to minimize the expected total ordering costs, inventory holding costs and shortages costs where demands are Poisson distributed. Because of the applicability to multi-location inventory systems, stochastic joint replenishment problem (SJRPs) is a challenging research area.

The problem of stochastic joint replenishment policies has been one of the most important issues faced especially by operations managers. Despite its practical importance, solution of the stochastic joint replenishment problems (SJRPs) is extremely hard. The existing policies in the literature do not dominate each other over the whole parameter space.

The joint replenishment problems (JRPs) are described as a single-location/multi-products system in the literature. However, it can also be described as a one product/multi-locations system where one product is supplied by a single warehouse, from which all the locations/retailers must replenish. The locations/retailers could replenish independently, but it may be beneficial to place their replenishments especially if the warehouse is far away. Consequently, the transportation costs will be reduced. For instance, if one considers the problem in a single-location/multi-products system, it might be debatable whether the lead times may differ, since the products are supplied by a single source. In the multi-locations/single-product setting, it is very likely that the lead times differ, since the times to reach the different locations naturally may differ (Nielsen and Larsen (2005)).

2.2 Classification of the Literature

In this study, we consider joint replenishment problems (JRPs) in a single-location/multi-item and single-item/multi-location inventory systems. The JRPs can be classified into two main categories according to the nature of demand; that is, deterministic JRPs and stochastic JRPs as shown in Figure 2.1. The deterministic JRPs have received a lot of attention in the literature (e.g., Aksoy, Y., Erenguc, S., 1988; Goyal S. and Satir A. T., 1989; and Khouja M. and Goyal

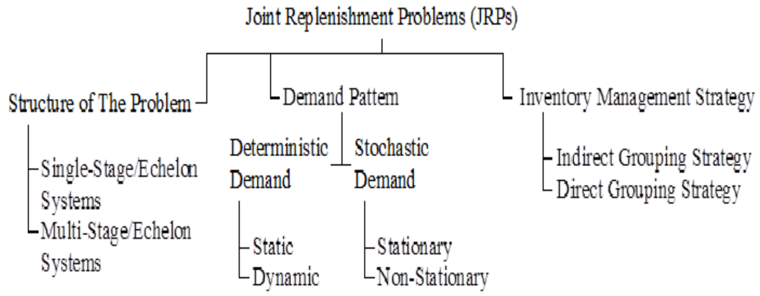

Figure 2.1: Classification of the Joint Replenishment Problems

S., 2008).

Deterministic JRPs could be divided into two main categories; namely, JRPs involving one type of operations (Ben-Daya and Hariga, 1995; Cha and Moon, 2005, 2006; Federgruen and Zheng, 1992; Frenk et al., 1999; Goyal and Deshmukh, 1993; Hariga, 1994; Hong, Kim, 2009; Hoque, 2006; Kaspi and Rosenblatt, 1985a, 1985b, 1991; Khouja et al, 2000; Khouja, 2005; Silver, 1976; van Eijs et al., 1992; Fung and Ma, 2001; Goyal, 1974a, 2002; Lee and Yao, 2003; Van Eijs, 1993; Viswanathan, 1996, 2002; Wildeman et al., 1997; Klein and Ventura, 1995; Olsen, 2003, 2005; Porras and Dekker, 2006; Queyranne, 1987; Xu et al., 2007) and JRPs involving several types of operations (Goyal, 1973a, 1973b, 1973c, 1974c, 1974b, 1975a, 1975b, 1976, 1977, 1980, 1987, 1988a, 1988b, 1988c; Goyal and Belton, 1979; Nocturne, 1973; Shu, 1971; Kaspi, 1991; Andres and Emmons, 1976; Graves, 1979).

In addition, JRPs involving one type of operations could be further divided into two subcategories depending on whether one considers a single buyer (e.g., Ben-Daya and Hariga, 1995; Federgruen and Zheng, 1992; Goyal and Deshmukh, 1993; Hariga, 1994; Honga, Kim, 2009; Kaspi and Rosenblatt, 1985; Kaspi and Rosenblatt, 1991; Silver, 1976; van Eijs et al., 1992; Fung and Ma, 2001; Goyal, 1974a; Goyal, 2002; Lee, Yao, 2003; Van Eijs, 1993; Viswanathan, 1996, 2002; Wildeman et al., 1997; Klein and Ventura, 1995; Olsen, 2003, 2005; Khouja, 2005; Frenk et al., 1999; Queyranne, 1987; Rosenblatt and Kaspi, 1985; Goyal, 1973a, 1973b, 1973c, 1974c, 1974b, 1975a, 1975b, 1976, 1980, 1988a, 1988b, 1988c; Goyal and Belton, 1979; Nocturne, 1973; Shu, 1971; Kaspi, 1991; Andres and Emmons, 1976) or multiple buyers (Chan et al., 2006; Chan et al., 2003; Li, 2004); note however that the single buyer case has been studied more often and related contributions could be further divided into those addressing the classical JRP (Ben-Daya and Hariga, 1995; Federgruen and Zheng, 1992; Goyal and Deshmukh, 1993; Hariga, 1994; Honga, Kim, 2009; Kaspi and Rosenblatt, 1985; Kaspi and Rosenblatt, 1991; Silver, 1976; van Eijs et al., 1992; Fung and Ma, 2001; Goyal, 1974a; Goyal, 2002; Lee, Yao, 2003; Van Eijs, 1993; Viswanathan, 1996, 2002; Wildeman et al., 1997; Klein and Ventura, 1995) and those addressing its generalizations (Olsen, 2003, 2005; Khouja, 2005; Frenk et al., 1999; Queyranne, 1987; Rosenblatt and Kaspi, 1985).

Recall that the classical JRP (C-JRP) assumes that both the joint replenishment ordering cost and the item-specific ordering costs are independent of the group of items jointly replenished. A generalization of the C-JRP consists of relaxing the assumption that item-specific ordering costs are independent of the group of items jointly replenished. Arkin et al. (1989) proved that the JRP is an NP-hard problem; therefore, it can not be solved by polynomial time algorithms.

The solution strategies to the JRP can be basically classified into two main categories: indirect grouping strategy (IGS) where a replenishment is made at regular time-cycles and each item has a replenishment quantity sufficient to last for exactly an integer multiple of the regular time cycle (e.g. Ben-Daya and Hariga, 1995; Federgruen and Zheng, 1992; Goyal and Deshmukh, 1993; Hariga, 1994; Honga, Kim, 2009; Kaspi and Rosenblatt, 1985; Kaspi and Rosenblatt, 1991; Silver, 1976; van Eijs et al., 1992; Fung and Ma, 2001; Goyal, 1974a; Goyal, 2002; Lee, Yao, 2003; Van Eijs, 1993; Viswanathan, 1996, 2002; Wildeman et al., 1997; Klein and Ventura, 1995; Olsen, 2003, 2005; Khouja, 2005; Frenk et al., 1999; Queyranne, 1987; Rosenblatt and Kaspi, 1985; Goyal, 1973a, 1973b, 1973c, 1974c, 1974b, 1975a, 1975b, 1976, 1980, 1988a, 1988b, 1988c; Goyal and Belton, 1979; Nocturne, 1973; Shu, 1971; Kaspi, 1991; Andres and Emmons, 1976), and direct grouping strategy (DGS) where items are partitioned into a predetermined number of groups and the items in each group are jointly replenished with the same cycle time (Aggarwal, 1984; Bastian, 1986; Chakravarty, 1981, 1985, Chakravarty et al., 1982, 1985; Olsen, 2003, 2005; Rosenblatt and Kaspi, 1985; Page and Paul, 1976; Queyranne, 1987 ; Queyranne and Sun, 1993; Schwarz, 1987). Groups in IGS are indirectly formed by products having the same integer multipliers. Van Eijs et al. (1992) reveal that IGS outperforms DGS for high major ordering cost because many products can be jointly replenished when using an IGS. Another classification according to the quality of the solutions, it can be classified into two main categories, namely, optimal solution (i.e., Goyal, 1974a; van Eijs, 1993; Viswanathan, 1996) and heuristic solution (i.e., Brown, 1967; Goyal, 1973; Hariga, 1994). For the optimal algorithms, there are two classes of cyclic policies for the JRPs, namely, a strict cyclic policy where there is at least one item ordered every replenishment order, and a general cyclic policy, where that condition is not necessarily satisfied.

For stochastic JRPs where demand is assumed to be stochastic and stationary, two main inventory control policies have been most commonly used in the literature . Recall that inventory control policies are concerned with decisions such as when often to place an order and how much to order. The literature reveals that these policies could be divided into two main categories depending on how often the inventory status should be known; namely, continuous review policies and periodic review policies; to be more specific, continuous review policies requires knowledge of inventory status at all times known (e.g., Balintify, 1964; Silver, 1965; Ignall, 1969; Renberg and Planche, 1967; Silver, 1974; Thompstone and Silver, 1975; Federgruen et al. 1984, Pantumsinchai, 1992; Melchiors, 2002; Nielsen and Larsen, 2005; and Johansen and Melchiors, 2003), whereas period review policies requires knowledge of inventory status at regular points in time only (e.g., Atkins and Iogyun, 1988; Viswanathan, 1997; and Ozkaya et al., 2006). The main inventory control policies for SJRPs can be considered as indirect grouping strategies, since the items are replenished on period basis.

2.3 Deterministic Joint Replenishment Problems (JRP)

Deterministic joint replenishment problem (DJRP) is a version of JRP when demand for each item is deterministic and constant. The objective is to minimize the total costs where the total cost consists of three types of costs; ordering cost, inventory holding cost. Two main strategies have been most commonly used in the literature for addressing the DJRP; namely, indirect grouping strategy and direct grouping strategy. The model formulations for the strategies and typical assumptions are discussed in this section.

2.3.1 Typical Assumptions of Deterministic JRP

The following assumptions for deterministic JRPs are made (see e.g., Van Eijs, 1993; Viswanathan, 1996, 2002; Wildeman et al., 1997).

1. For each item, the relevant parameters are deterministic and constant over a planning horizon that is considered infinite for planning purposes; namely, demand rate, individual ordering cost, inventory holding cost;
2. For each item, replenishment lead time is zero or negligible;
3. For each item, shortages are not allowed;
4. The joint replenishment of a subset of products or all products achieves some sort of economies of scale and the joint ordering cost is independent of both the items ordered and their number;
5. Purchase orders are placed at equal time intervals and each item is replenished at equal time intervals;
6. Supply is readily available.

2.3.2 Indirect Grouping Strategy

Under an indirect grouping strategy, a replenishment order is made at fixed periods. The replenishment order cycle (the time between the placing of two successive replenishment orders) of each item is an integer multiple of this basic period. The problem is to determine the basic period F and the integer multiple m_i of each item i. This is also known as the basic cycle strategy.

Notations:

Parameters for indirect grouping strategy JRPs:

n : number of items,

d_i : demand/requirements of item i expressed in units per year,

h_i : cost of inventory holding of item i expressed in \$/unit of item per year,

A : joint ordering cost associated with each replenishment order (\$/order),

a_i : specific-item ordering cost of item i, incurred if item i is ordered in a replenishment order expressed in \$/order of item i.

Decision Variables for indirect grouping strategy:

F : basic period between successive replenishment orders, is a continuous variable,

m_i : an integer time multiple of the basic period F for the i^{th} item in a planning period, say, 1 year,

The JRP and its variants are commonly addressed using a cyclic scheduling approach, where one assumes that the cycle time of each item, say T_i, is an integer multiple, say m_i, of a basic period, say F; that is, $T_i = m_i F$. The order quantity for item i is $R_i = T_i d_i = m_i F d_i$. For example, Item 1 and 3 are ordered every basic period (such as a day or a week) F an order is placed (that is, $m_1 = 1$, $m_3 = 1$). Item 2 is ordered every second replenishment order of the items, with a replenishment quantity sufficient to last a time interval of duration $2F$, so that $m_2 = 2$. Each time it will be replenished just its stock hits the zero level. We wish

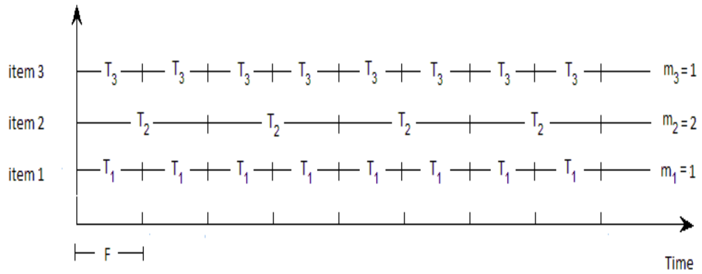

Figure 2.2: Evolution of Replenishment Process

to select the values of F and the m_i's to keep the total costs as low as possible. This is illustrated in Figure (2.2). The objective is to minimize the total cost which consists of ordering costs and holding costs, that is,

$$TC(F, m_1, ..., m_n) = \frac{1}{F}\Big[A + \sum_{i=1}^{n} \frac{a_i}{m_i}\Big] + \frac{F}{2}\sum_{i=1}^{n} h_i d_i m_i, \tag{2.1}$$

where F is a continuous variable and $m_i \geq 1$, $i = 1, 2, ..., n$, are integers.
By taking the partial derivative of (2.1) with respect to basic period, F, the optimal basic period F^* is obtained

$$F^* = \sqrt{2\Big(A + \sum_{i=1}^{n} \frac{a_i}{m_i}\Big)\Big/\Big(\sum_{i=1}^{n} h_i d_i m_i\Big)}, \tag{2.2}$$

substituting the value of F^* into (2.1), the optimal total cost,

$$TC^* = \sqrt{2\Big(A + \sum_{i=1}^{n} \frac{a_i}{m_i}\Big) \cdot \Big(\sum_{i=1}^{n} h_i d_i m_i\Big)}, \tag{2.3}$$

for any value of basic period F, the optimal value of the time multiple for item i, say m_i, must satisfy the condition (Goyal, 1973),

$$\sqrt{m_i(m_i-1)} \leq T_i^*/F \leq \sqrt{m_i(m_i+1)} \tag{2.4}$$

However, the vector of time multiples can not be determined without knowing the basic period, F and vice versa.

2.3.3 Direct Grouping Strategy

Under a direct grouping strategy, items are replenished into different groups which are ordered independently and each group has its own basic period. This is also known as a fixed cycle strategy.

Notations:

Parameters for direct grouping strategy JRPs:

n : number of items,

m : number of groups,

k : group number, $k = 1, ..., m$,

n_k : number of items in a group k,

d_i : demand/requirements of item i expressed in units per year,

h_i : cost of inventory holding of item i expressed in \$/unit of item per year,

A : joint ordering cost associated with each group (\$/group),

a_i : specific-item ordering cost of item i, incurred if item i is ordered in a group, expressed in \$/order of item i.

Decision Variables for direct grouping strategy:

F_k : replenishment order cycle time between ordering items in group k.

The ordering costs and the inventory holding costs per unit time for the k^{th}

group of n_k items can be written as,

$$TC_k = \frac{n_k A}{F_k} + \frac{F_k}{2} \sum_{i \in k} h_i d_i, \tag{2.5}$$

the total cost,

$$TC_k = \sum_k TC_k, \tag{2.6}$$

By taking the partial derivative of (2.5) with respect to F_k^*, the optimal common cycle for the k^{th} group is obtained,

$$F_k^* = \sqrt{2n_k A \Big/ \sum_{i \in k} h_i d_i}, \tag{2.7}$$

Hence, the optimal quantity order for item i in the k^{th} group,

$$R_i^* = d_i F_k^*, i \in k,$$

substituting the value of F_k^* into (2.5) gives the optimal total cost,

$$TC_* = \sqrt{2n_k A \sum_{i \in k} h_i d_i}, \tag{2.8}$$

The problem is to divide the number of items into a predetermined number of groups so as to minimize the total cost. The key difference between indirect grouping strategy and direct grouping strategy is that the replenishment order cycles of the groups modelled by indirect grouping strategy are multiple integers of some basic cycle, where that is not the case for groups modelled by direct grouping (Van Ejis et al., 1992). The challenging issue of direct grouping strategies is to divide the number of items into a certain number of different groups.

2.3.4 Critical Analysis of The Literature

During the last three decades, a number of heuristic methods have been proposed to address the JRPs and to obtain the near optimal solution. The heuristics can be classified into two classes; namely iterative procedures (e.g., Brown, 1967; Goyal, 1973, 1974b, 1988b, 1988c) and non-iterative procedures (e.g., Silver, 1976; Goyal and Belton, 1979; Kaspi and Rosenblatt, 1983). Brown (1967) is an early author who suggested an iterative method to determine the vector of time multiples, say , where an initial solution is when all items are ordered every period. Goyal (1974a, 1975) proposes an enumeration algorithm to obtain the optimal solution. However, Andres and Emmons (1976) show that Goyal (1974a) does not guarantee the optimal solution, and proposed a modified version with the modified lower bound of the basic period . Goyal (1978, 1988a) Modified Goyal (1974a) where the lower and the upper bounds on the basic period are sharpened. Klein and Ventura (1995) is a discrete counterpart of Goyal (1974a). This method, however, does not guarantee an optimal solution as the authors use the same bounds as Goyal (1974a). One weakness of these algorithms is computationally prohibitive for large problems. Some optimal algorithms for the classic deterministic JRPs are presented in appendix A.

Silver (1976) addresses the classical JRP by proposing a simple heuristic for its solution. Goyal and Belton (1979) modify Silver (1976). The only difference between Silver (1976) and Goyal and Belton (1979) is that the latter adjusts Silvers criterion for ranking items by taking account of the joint ordering cost. Kaspi and Rosenblatt (1983) make further improvement on Silver (1976). Kaspi and Rosenblatt (1985) propose a solution procedure for the JRP that combines the procedures proprosed by Goyal (1974b) and Silver (1976). The initial value of F is obtained using Silver (1976) and then Goyal (1974b) is implemented. Goyal (1988b, 1988c) and Kaspi (1991) propose a modification of Silver's method with a different initial value of F.

Kaspi and Rosenblatt (1991) suggest a simple procedure referred to as RAND, for determining the economic ordering quantity for items jointly replenished. RAND improves Kaspi and Rosenblatt (1983) by determining the lower and upper bounds of the basic period. Goyal and Deshmukh (1993) propose a modified version of RAND by improving the lower bound of the basic period. Hariga (1994) proposes two improvement heuristics for the JRP where the initial solution is obtained by relaxing the requirement that the cycle time of each item is an integer multiple of the basic period. The goal of the first heuristic is to generate the time multiple that make the cycle time of each item converge to the solution of relaxed JRPs (RJRP). The second heuristic is a modified version of Goyal (1973) where the initial values of time multiples are obtained using RJRP.

As can be understood from the above, not all heuristics can guarantee the optimal solution. However, there are a few procedures that guarantee the optimal solution. Van Eijs (1993) shows that the algorithm of Goyal (1974a) does not always produce an optimal cyclic policy and proposes a modified version that does. Goyal (1974a) and Van Eijs (1993) require much time to obtain the optimal solution. Viswanathan (1996) proposes an algorithm where the lower and upper bounds on the basic period are sharpened to reduce the computational time. Goyal (1974a) performs better when the joint replenishment cost is small, van Eijs (1993) performs better when the joint replenishment cost is high, and Viswanathan (1996) performs better when the joint replenishment cost is moderate. Wildeman et al. (1997) propose an optimal approach to the JRPs using a dynamic Lipschitz constant and producing a solution in less time that is more efficient than other optimal algorithms. Fung and Ma (2001) propose two procedures for JRPs where the lower and upper bounds on the basic period are sharpened. The first procedure of Fung and Ma (2001) performs better when the joint replenishment cost is small, whereas the second procedure performs better for any type of joint replenishment cost. Viswanathan (2002) shows that the algorithm of Fung and Ma (2001) does not always produce a strict cyclic policy and proposed a modified version that does. Also, Viswanathan (2002) shows Viswanathan (1996) is computationally more efficient than other algorithms.

2.4 Stochastic Joint Replenishment Problems (SJRP)

A stochastic joint replenishment problem (SJRP) is a version of JRP when demand for each item is stochastic and stationary. The objective is to minimize the expected total costs per costs per unit time where the expected total cost consists of three costs; expected ordering cost, expected inventory cost and expected shortage cost. Two main policies have been most commonly used in the literature for addressing the SJRP; namely, continuous review policies and periodic review policies.

2.4.1 Critical Analysis of The Literature

Although the stochastic joint replenishment problem (SJRP) is a critical issue in manufacturing and distribution systems in single echelon/stage systems, a good solution for this problem is extremely difficult. As the stochastic joint replenishment problem (SJRP) is different from its deterministic version of JRP in terms of the nature of demand, therefore a number of heuristic algorithms have been proposed to address the SJRP and to obtain the near-optimal solution. The SJRP policies are reviewed in this section are listed in Table 2.1.

Balintify (1964) is the first author who proposes a policy to address the SJRP, which is denoted by (s, c, S) policy, also known as the can-order policy. The principle of the (s, c, S) policy is that when the inventory position (on-hand inventory plus inventory on-order minus backorders) of any item drops to or below its s, an order is placed to bring up to its level S, at the same time, any other item with an inventory position drops or below its c, is also included in the replenishment. Silver (1965) addresses a two-item problem where items have identical costs where demand is a Poisson generated with rate λ and is a one unit at time. Ignall (1969) also deals with the same problem, and obtains an optimal policy. However, the optimal solution obtained for (s, c, S) policy cannot be generalised for other problems and computationally intractable. Silver (1973) proposes three approaches to obtain the same total cost function of the problem when demand is Poisson and replenishment lead time is zero. Silver (1974) also considers the same problem with constant replenishment lead time and shows that by using the (s, c, S) policy, it is likely to obtain substantial cost savings in comparison with an independent single-item (s, S) policy. Federgruen et al. (1984) suggest a semi-Markov decision model and use a decomposition approach similar to Silver (1974). They focus on calculating the control parameters of the (s, c, S) policy and propose a heuristic method using a policy-iteration algorithm to find the control parameters. Schultz and Johansen (1999) reveal that it is hard to find an exact optimal (s, c, S) policy. Melchiors (2002) proposes a new compensation approach for improving the solution to the (s, c, S) policy when demand is Poisson generated. However, the approximations used need extensive iterative computations and may result in substantial deviations from simulated costs in some cases. Johansen and Melchiors (2003) propose a periodic review version of (s, c, S) policy which performs well when demand is irregular . As all policies focus on how to develop the optimal (s, c, S) parameters based on their proposed algorithms, they are realistically difficult to apply to large size problems.

Authors	Year	SJRP Policy	Review Type	Description	Limitations	Relevant Chapter
Federgruen, Groenevelt & Tijms	1984	(s, c, S)	Continuous	Reorder point Can-order point Order-up-to level	May not synchronize ordering of heterogeneuos items Parameters diffiicult to compute	-
Atkins & Iyogun	1988	(F, S)	Periodic	Fixed order interval Synchronizes ordering	F and S are independent	3
Atkins & Iyogun	1988	(mF, S)	Periodic	Order interval varies by items	F and S are independent	3
Pantumsinchai	1992	(Q, S)	Continuous	Joint reorder point	May not order when only a few items are short	4
Viswanathan	1997	(F, s, S)	Periodic	Reorder point order-up-to level	Does not synchronize transportation with replenishment	3, 4
Nielsen & Larsen	2005	(Q, s, S)	Continuous	Joint reorder point item reorder point ordr-up-to level	Does not synchronize transportation with replenishment	4

Table 2.1: Joint Replenishment Plocies in the Literature

Atkins and Iyogun (1988) propose two periodic review policies where all items are ordered up to their level S periodically. In the first policy, all items are ordered up to their levels S and the second policy is a modified version of the first policy where items belonging to a base set are ordered every F time units and other items are ordered every mF time units where m is a positive integer. Pantumsinchai (1992) compare the performance of (s, c, S) policy with (mF, S) policy proposed by Atkins and Iyogun (1988), and the (Q, S) policy proposed by Renberg and Planche (1967) for Poisson demand process for items. In the latter policy, they use a group reorder point as mechanism to place an order. The principle of the policy is that the inventory is reviewed only when the aggregate demand since the last order reaches Q, all items are ordered up to their levels S. Pantumsinchai (1992) shows that (s, c, S) policy performs well when the joint order cost is low, and the (Q, S) policy and (mF, S) policy perform well when the joint order cost is high, and shortage costs are low.

Zheng and Federgruen (1991) consider an (s, S) policy for a single item and develop an efficient algorithm to compute the optimal S and s values. Viswanathan (1997) propose a periodic review policy, denoted by (F, s, S), where the inventory is reviewed periodically every F time units and all items with inventory positions below their s are ordered up to S. in this policy, the optimal s, S for each item and F are the decision variables. For a given F, the optimal (s, S) policy for the n items are computed using Zheng and Federgruen (1991) algorithm. The initial value of F is obtained by solving the deterministic JRPs, and then search for the best F in either direction until no further improvement is made.

Nielsen and Larsen (2005) propose a continuous review policy referred to as (Q, s, S) which is originally suggested by Viswanathan (1997). In this policy, the inventory is reviewed only when the aggregate demand since the last order reaches Q, and any item drops to or below its s, an order is placed to bring up to its level S. They used Markov decision theory to develop an analytical solution procedure to evaluate the costs of a particular improvement of the policy under a Poisson demand process. The analytical solution procedure is used to develop a method to compute the optimal (Q, s, S) policy. The problem is decomposed into n single-item problems and structured as a semi-Markov decision problem. The algorithm has two loops in which Q is varied and the other loop in which Zheng and Federgruen (1991) algorithm is applied to compute the optimal (s, S) for each item for a given Q.

Özkaya et al. (2006) proposed a $(T|Q, S)$ policy which is a hybrid of the continuous review (Q, S) policy, proposed by Renberg and Planche (1967), and the periodic review (F, S) policy of Atkins and Iyogun (1988). Under this policy, the inventory is reviewed only when the aggregate demand since the last order reaches Q, or F time units have elapsed, whichever occurs first, bring the inventory positions of all items up to their levels S. The total cost expression was derived under a Poisson demand process for each item and under compound demand process where items have stochastic size demands that arrive according to a Poisson process. They claim that their own policy outperforms all existing policies, but their policy is basically a minor modification of the (Q, S) policy proposed by Renberg and Planche (1967) and (F, S) policy of Atkins and Iyogun (1988). There are some concerns about their numerical results (Larsen, 2008).

As the above discussion of the replenishment control policies for SJRP in literature demonstrates that the stochastic joint replenishment problem is a wide research venue for the development of more efficient computational algorithms and replenishment control policies. Therefore, the main contribution of this research is to design an algorithm for improving the solution of the (mF, S) policy, proposed by Atkins and Iyogun (1988) by solving the deterministic version of JRP and choosing the best integer multiple for each item and improving the initial review period by an increment of 0.01 in either direction until there is no further improvement in cost. Also, we propose another periodic review policy for SJRP, which is referred to as (mF, s, S) policy. The principle of this policy is that the inventory is reviewed periodically every F time units and all items with inventory positions below their s are ordered up to S every mF time units, where m is a positive integer.

Moreover, a new periodic review policy, denoted (F, Q, S) is proposed. The principle of the policy is that the inventory is reviewed every F unit time and all items are ordered up to their levels S only when the aggregate demand during a replenishment order cycle reaches Q. This policy combines features of both periodic and continuous review policies into an effective policy. Consequently, it attempts to exploit the benefits of two separate policies. As a result, it reduces to these two policies in the limit; as $F \to 0$, we obtain the (Q, S) policy, and as $Q = 1$, we obtain the (F, S) policy. For the (F, Q, S) policy, we develop expressions for the operating characteristics of the inventory system and constructed the expected total cost function for Poisson demand process. Numerical results have been conducted to study the sensitivity of the policy to various system parameters and to evaluate the performance of the proposed policy over existing policies. Another new proposed policy is to add a must-order point for each item in the (F, Q, S) policy. This policy is denoted by (F, Q, s, S). For the latter, we believe that the robustness of the (F, Q, S) policy performance will be increased. Moreover, the review periodic policies enormously ease implementation in practice.

Chapter 3

Periodic Review (mF, s, S) Policy

This chapter is concerned with the stochastic joint replenishment problem (SJRP) where demand that cannot be satisfied immediately is backordered. We propose a new periodic review policy, called (mF, s, S) policy. The proposed policy is a generalisation of (F, s, S) policy by assuming that the review period of each product is an integer multiple of a basic period and restricts these multiples to the value m where m is a positive integer. We derive closed form expressions for the expected total cost for (mF, s, S). In addition, heuristic Algorithms have been developed to search for near optimal parameters for the (mF, s, S) policy as well as another heuristic algorithm has been developed for an (mF, S) policy. Numerical tests against literature benchmarks have been presented.

This chapter is organised as follows. In section 3.1 we present a brief literature review for periodic review policies, introduce a new periodic review policy for SJRP and discuss its importance. In section 3.2 we present a mathematical formulation of SJRP as well as typical assumptions and notations. A Markov decision process for the single-item inventory model formulation is discussed and then we present heuristic algorithms to solve SJRP in Sections 3.3 and 3.4 respectively. In section 3.5 we provide comparative analysis and numerical results for the periodic review policies to help evaluate their performance. Finally, a conclusion is presented in section 3.6.

3.1 Introduction

We consider a stochastic joint replenishment problem (SJRP) where demand that cannot be satisfied immediately is backordered, so as to minimise the expected total ordering, inventory holding and shortages costs per unit time. A new periodic review policy, called (mF, s, S) policy is proposed for stochastic JRP. In this policy, the inventory position for an item i is reviewed periodically every m_iF time units and if the inventory position is below their s_i, then bring the inventory position up to level S_i, where m_i is a positive integer.

Herein we discuss both situations where the inventory holding costs are higher than the shortages costs or lower. In practice, the first situation where the inventory holding costs are higher than the shortages costs is applicable where the firm has a monopoly power. The customers do not have an alternative place to make their purchases and must wait until their order is fulfilled. Some examples in monopolistic industries are automobile companies and mobile companies.

Atkins and Iygun (1988) suggest two periodic review policies namely, (F, S), and (mF, S) where the items are replenished periodically irrespective of their inventory positions. Another periodic review policy, (F, s, S) is suggested by Viswanathan (1997). Under the (F, s, S) policy, inventory of each item is reviewed at a constant time period. An independent, periodic review (s, S) policy is used for each item. The optimal (s, S) policy for each item is computed assuming that the item bears only the minor setup cost. In (F, s, S) policy, the review period F is a decision variable and must be the same for all items. However, the (F, s, S) policy is weak when the inventory holding costs are higher than the shortages costs. As each item has a typical replenishment order cycle, therefore, we need to avoid the shortages and reduce the number of periods without stock. In the case where the inventory holding costs are lower than the shortages costs, it might be optimal to replenish at every opportunity to avoid the shortages costs.

Moreover, the (mF, s, S) policy combines the features of the (F, s, S) policy and

(mF, S) policy. Also, The (mF, s, S) policy is a generalisation of (F, s, S) policy by assuming that the review period of each product is an integer multiple of a basic period and restricts these multiples to the value m_i for each item i where m_i is a positive integer. As a result, (F, s, S) policy is a special case of (mF, s, S) policy when $m = 1$ for each item, and (mF, S) policy is a special case of (mF, s, S) policy when the reorder point for each item $s = S - 1$.

3.2 Model Formulation and Assumptions for Stochastic JRP

The model formulations and typical assumptions are discussed in this section.

3.2.1 Typical Assumptions of Stochastic JRP

The following assumptions for stochastic JRPs are made (see e.g., Goyal and Satir, 1989) and also the notation to be used is defined in this section.

1. For each item, demand is assumed to be independent of the other item, stochastic and stationary;

2. For each item, shortages are allowed and are assumed to be backorders, where a customer agrees to wait until their order is fulfilled;

3. For each item, the other relevant parameters are deterministic and constant over a planning horizon that is considered infinite for planning purposes; namely, replenishment lead time, specific-item ordering cost, inventory holding cost, and shortages costs;

4. The joint replenishment of a subset of products or all products achieves some sort of economies of scale and the joint ordering cost is independent of both the items ordered and their number;

5. Supply is readily available.

Notations:

Parameters

n : number of items,

λ_i : mean of Poisson distribution of demand for item i expressed in units per year,

L_i : replenishment lead time of item i expressed in unit of time per year,

h_i : cost of inventory holding of item i expressed in \$/unit per year,

p_i : backorder cost per unit and time unit for item i,

π_i : one-off shortage cost of per unit of item i,

A : joint ordering cost associated with each replenishment order (\$/order),

a_i : specific-item ordering cost of item i, incurred if item i is ordered in a replenishment order expressed in \$/order of item i.

Decision Variables

F : basic review period length for replenishment expressed in unit of time per year,

m_i : the integer time multiple of the basic review period F for item i,

T_i : review period for item i expressed in unit of time per year, where $T_i = m_i F$,

s_i : reorder point for item i,

S_i : order up-to level for item i.

3.2.2 The Decomposition Approach

First we decompose the problem into n single-item inventory problems. After that we structure the single-item problem. For a given review period F, and a given m_i, the optimal (s_i, S_i) policy for each item i can be computed using the algorithm of Zheng and Federgruen (1991) with a review period $T_i = m_i F$. The expected total cost (TC(F)) of the $(m_i F, s_i, S_i)$ policy is approximately:

$$TC(F) = \frac{A}{F} + \sum_i C_i^*(m_i F) \tag{3.1}$$

where $C_i^*(m_iF)$ is the cost per unit time of the optimal (s_i, S_i) policy for item i corresponding to a review period of T_i, and item i only bears the item-specific ordering cost. The actual cost of (m_iF, s_i, S_i) policy is less than that evaluated by expression (3.1), since the major setup cost A is not incurred in a review period in which none of the items is ordered.

3.3 Single-item Model Formulation

In this section, we consider the case of a single product problem and to simplify the notation let us drop the product index i. First we formulate the inventory holding and shortages costs function for one period. Then we show how the cost expression can be derived in the case where demands are generated by Poisson processes and units are demanded one a time.

3.3.1 Inventory Holding and Shortages Costs Function for one period

Consider Figure 3.1 and an arbitrary review time t and the replenishment lead time L is a constant. the next review takes place at time $t + T$. Note that everything on order immediately after the review at time t will arrive by $t+L$ but nothing not on order can arrive before time $t+L+T$. Let $IP(t)$ be the inventory position (on-hand inventory plus inventory on-order minus backorders) at time t, D(t,t+z) be the stochastic demand in the interval (t,t+z), and $IL(t + z)$ be the inventory level (on-hand inventory minus backorders) at time $t+z$. Consequently we have the simple relationship which is true for any time z between $t + L$ and $t + L + T$ is

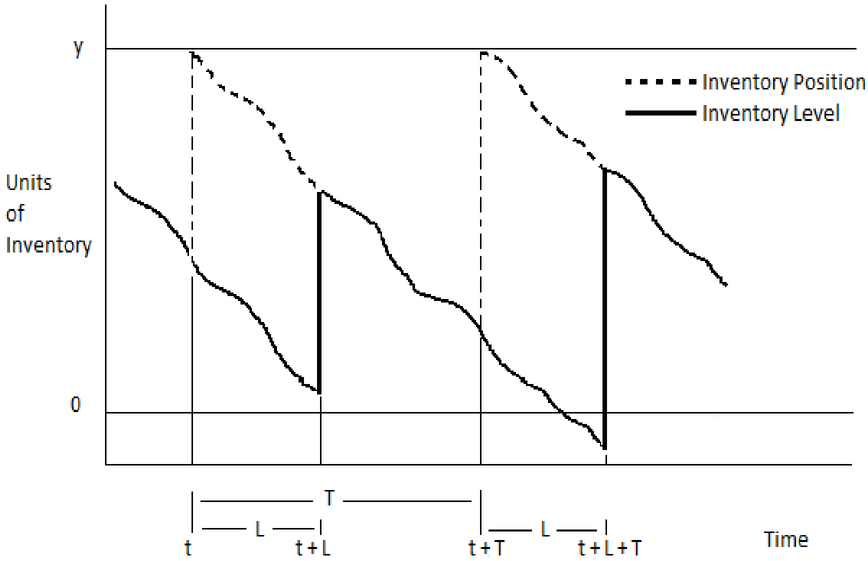

Figure 3.1: Evolution of the Inventory Position and the Inventory Level for Periodic Review

$$IL(t+z) = IP(t) - D(t, t+z), \quad z \leq T \tag{3.2}$$

Now, we obtain the the distribution of the inventory level at time $t+z$. Assume that the inventory position $IP(t)$ of the system is y immediately after the review at time t. The inventory level $IL(t+z)$ at time $t+z$ is the inventory position $IP(t)$ at time t minus the cumulative demand during the interval $[t, t+z]$. Thus, the distribution of the inventory level $P(IL(t+z))$ can be obtained as

$$P(IL(t+z) = k) = P(D(t, t+z) = y-k), \quad k \leq y \tag{3.3}$$

After obtaining the distribution of the inventory level from (3.3) we can derive the inventory holding and shortages costs function. Let us use the notation

$$\begin{aligned} (x)^+ &= \max(x, 0) \\ (x)^- &= \max(-x, 0) \end{aligned}$$

Given the inventory level $IL(z)$ at time $z \in [t+L, t+L+T]$, the expexted inventory holding cost per unit per unit of time at time z is $h\int_L^{L+T} E(IL(z)^+)dz$, the expected backorder cost per unit per unit of time at time z is $p\int_L^{L+T} E(IL(z)^-)dz$ and the expected one-off shortags cost per unit at time z is $\pi\Big[E(IL(t+L+T)^-)-E(IL(t+L)^-)\Big]$. Consequently, if the inventory position at time t is y, the expected number of on hand inventory held H(y) at any time $t+z$ between $t+L$ and $t+L+T$ is

$$
\begin{aligned}
H(y) &= \int_L^{L+T}\Big[\sum_{k=0}^{y} kP(IL(z)=k)\Big]dz \\
&= \int_L^{L+T}\Big[\sum_{k=0}^{y} kP(D(t,z)=y-k)\Big]dz \\
&= \int_L^{L+T}\Big[\sum_{k=0}^{y} (y-k)p_{\lambda z}(k)\Big]dz \qquad (3.4)
\end{aligned}
$$

where $p_{\lambda z}(k)$ is the probability of the demand during a period of length z, That is, Poisson distributed with parameter λz $(k = 0, 1, 2, ...)$. It is easy to develop the expression (3.4) by using the properties of Poisson distribution (Appendix B). Thus,

$$
\begin{aligned}
H(y) &= \int_L^{L+T}\left[\sum_{k=0}^{y}(y-k)p_{\lambda z}(k)\right]dz \\
&= \int_L^{L+T}\left[\sum_{k=0}^{\infty}(y-k)p_{\lambda z}(k) - \sum_{k=y}^{\infty}(y-k)p_{\lambda z}(k)\right]dz \\
&= \int_L^{L+T}\Big[y-\lambda z\Big]dz + \int_L^{L+T}\Big[\sum_{k=y}^{\infty}(k-y)p_{\lambda z}(k)\Big]dz \\
&= \left[yz - \frac{\lambda z^2}{2}\right|_L^{L+T} + \int_L^{L+T}\Big[\sum_{k=y}^{\infty}(k-y)p_{\lambda z}(k)\Big]dz \\
&= T\Big(y-\lambda L-\frac{\lambda T}{2}\Big) + \int_L^{L+T}\Big[\sum_{k=y}^{\infty}(k-y)p_{\lambda z}(k)\Big]dz \qquad (3.5)
\end{aligned}
$$

For backorder cost, defined as the cost per unit short per time unit, the backorder cost depends not only on the number of units short but the time the customer must wait for delivery. Thus, we compute the expected number of backorders that occur between $t+L$ and $t+L+T$ for the reason that everything on order immediately after the review at time t will arrive in the system by time $t+L$ but nothing not on order can arrive before time $t+L+T$.Therefore, if the inventory position at time t is y, the expected number of backorders $B(y)$ of unit time incurred from $t+L$ to $t+L+T$ is

$$\begin{aligned} B(y) &= \int_{L}^{L+T} \Big[\sum_{k=y}^{\infty} -kP(IL(z)=k)\Big]dz \\ &= \int_{L}^{L+T} \Big[\sum_{k=y}^{\infty} -kP(D(t,z)=y-k)\Big]dz \\ &= \int_{L}^{L+T} \Big[\sum_{k=y}^{\infty} (k-y)p_{\lambda z}(k)\Big]dz \qquad (3.6) \end{aligned}$$

from (C.4) (see Appendix B), we have

$$\begin{aligned} \sum_{k=y}^{\infty}(k-y)p_{\lambda z}(k) &= \sum_{k=y}^{\infty} kp_{\lambda z}(k) - \sum_{k=y}^{\infty} yp_{\lambda z}(k) \\ &= \lambda z \sum_{k=y-1}^{\infty} p_{\lambda z}(k) - \sum_{k=y}^{\infty} yp_{\lambda z}(k) \end{aligned}$$

The expected number of backorders $B(y)$ can be altered to a form which is easy to compute, that is

$$B(y) = \int_{L}^{L+T} \Big[\lambda z \sum_{k=y-1}^{\infty} p_{\lambda z}(k) - y\sum_{k=y}^{\infty} p_{\lambda z}(k)\Big]dz \qquad (3.7)$$

However, we can then compute the integral using the Poisson properties. From (C.7) and (C.8), we have

$$\begin{aligned} B(y) &= \frac{\lambda}{2}\Big[(L+T)^2 \sum_{k=y-1}^{\infty} p_{\lambda(L+T)}(k) - L^2 \sum_{k=y-1}^{\infty} p_{\lambda(L)}(k)\Big] \\ &\quad -\frac{y(y+1)}{2\lambda}\Big[\sum_{k=y+1}^{\infty} p_{\lambda(L+T)}(k) - \sum_{k=y+1}^{\infty} p_{\lambda(L)}(k)\Big] \\ &\quad -y\Big[(L+T)\sum_{k=y}^{\infty} p_{\lambda(L+T)}(k) - L\sum_{k=y}^{\infty} p_{\lambda(L)}(k)\Big] \qquad (3.8) \end{aligned}$$

For shortage cost, defined as the one-off penalty for failing to meet demand when it arises. Thus, if the inventory position at time t is yThe expected number of shortages that occur from $t+L$ to $t+L+T$ equals the number of shortages at $t+L+T$ (before arrival of any order) minus the number of shortages at $t+L$ (after arrival of any order). The expected number of shortages $S(y)$ at time $t+L+T$ is

$$S(y) \quad = \quad \sum_{k=y}^{\infty}(k-y)p_{\lambda(L+T)}(k) \tag{3.9}$$

and at time $t+L$

$$S(y) \quad = \quad \sum_{k=y}^{\infty}(k-y)p_{\lambda L}(k) \tag{3.10}$$

Thus, the expected number of shortages $S(y)$ incurred between time $t+L$ and $t+L+T$ is

$$S(y) \quad = \quad \sum_{k=y}^{\infty}(k-y)\Big[p_{\lambda(L+T)}(k) - p_{\lambda L}(k)\Big] \tag{3.11}$$

Then from using (C.4), we can write $S(y)$ as

$$\begin{aligned} S(y) \quad = \quad & \lambda(L+T)\sum_{j=y-1}^{\infty} p_{\lambda(L+T)}(k) - y\sum_{k=y}^{\infty} p_{\lambda(L+T)}(k) \\ & -\lambda L\sum_{k=y-1}^{\infty} p_{\lambda(L)}(k) - y\sum_{k=y}^{\infty} p_{\lambda(L)}(k) \end{aligned} \tag{3.12}$$

Then, the inventory holding and shortages costs function $G(y)$ can be obtained as

$$G(y) \quad = \quad hT\Big(y - \lambda L - \frac{\lambda T}{2}\Big) + (h+p)B(y) + \pi S(y) \tag{3.13}$$

3.3.2 Single-item Cost Function

Under an (T, s, S) policy, an item's inventory position follows a regenerative process, and hence has a steady state distribution. Each time placing an order and a new replenishment cycle begins, the item inventory position is brought up to level S and we have a regeneration. Note that we place an order when the inventory position drops or below to s. However, the time between the placing of two successive replenishments (i.e., the length of a cycle) will be an integer multiple of the basic period T. We assume that the cumulative demand Y_j for a specific item until j^{th} period, where $j = 1, 2, ...$ The demands in periods are independent and identically distributed (iid). Let $p_{\lambda T}(k)$ be the probability that k units will be demanded in one period. Then, since the demands in different periods are assumed to be independent, $p_{\lambda jT}(k)$ is the probability that k units are demanded in j periods. Let $N(S-s)$ be a random variable that represents the first period in which the cumulative demand exceeds $S-s$.

$$N(S-s) \quad = \quad \arg\min \{j | Y_j \geq S-s\} \tag{3.14}$$

Let,

$G(y)$	$=$	the expected cost of inventory holding and shortages function for a period of length T, where an inventory position is y at the beggining of the period; y integer,
$Z(S)$	$=$	the expected cost when the inventory position $IP = S$ at the beggining of a replenishement cycle,
$m(k)$	$=$	the expected number of periods until a cumulative demand exceeds k during a replenishment cycle, where $k = 0, 1, ..., S-s-1$
$M(S-s)$	$=$	the expected total number of periods until a cumulative demand exceeds $S-s$ during a replenishment cycle,

Hence, by the Proposition (5.9) of the regenerative processes in Ross (1970), we have that

$$C(s, S) \quad = \quad \frac{Z(S)}{FM(S-s)} \tag{3.15}$$

We need to obtain the expressions for the denominator and the numerator in (3.15). First, we compute the numerator $M(S-s)$ using the definition of the expected value (see Appendix B)

$$\begin{aligned}
M(S-s) &= \mathbb{E}(N(S-s)) \\
&= \sum_{j=0}^{\infty} P(N(S-s) > j) \\
&= \sum_{j=0}^{\infty} \Big(1 - P(Y_j \geq S-s)\Big) \\
&= \sum_{j=0}^{\infty} P(Y_j < S-s) \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{S-s-1} P(Y_j = k) \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{S-s-1} p_{\lambda jT}(k) \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{S-s-1} p^{j}_{\lambda T}(k) \\
&= \sum_{k=0}^{S-s-1} m(k) \qquad (3.16)
\end{aligned}$$

where $m(k) = \sum_{j=0}^{\infty} p_{\lambda T}^{j}(k)$, and $p_{\lambda T}^{j}(k)$ is the j-fold convolution of $p_{\lambda T}(k)$. However, $m(k)$ can then be derived easily as

$$\begin{aligned}
m(k) &= \sum_{j=0}^{\infty} p_{\lambda T}^{j}(k) \\
&= p_{\lambda T}^{0}(k) + \sum_{j=0}^{\infty} p_{\lambda T}^{j+1}(k) \\
&= p_{\lambda T}^{0}(k) + \sum_{j=0}^{\infty} p_{\lambda T}^{j+1}(k) \\
&= p_{\lambda T}^{0}(k) + \sum_{j=0}^{\infty} p_{\lambda T}(k) p_{\lambda T}^{j}(k) \\
&= p_{\lambda T}^{0}(k) + \sum_{j=0}^{\infty} \sum_{l=0}^{k} p_{\lambda T}(l) p_{\lambda T}^{j}(k-l) \\
&= p_{\lambda T}^{0}(k) + \sum_{l=0}^{k} p_{\lambda T}(l) m(k-l) \qquad (3.17)
\end{aligned}$$

The equation (3.17) can be solved recursively for $k = 0, 1, 2, ..., S-s-1$ as follows For $k = 0$, we have

$$\begin{aligned}
m(0) &= p_{\lambda T}^{0}(0) + p_{\lambda T}(0) m(0) \\
&= p_{\lambda T}^{0}(0) + p_{\lambda T}(0) m(0) \\
&= p_{\lambda T}^{0}(0) + p_{\lambda T}(0) m(0)
\end{aligned}$$

since $p_{\lambda T}^{0}(0) = 1$ and $p_{\lambda T}^{0}(k) = 0$ for any $k > 0$, then

$$m(0) = \frac{1}{1 - p_{\lambda T}(0)}$$

and for any $k > 0$,

$$m(k) = \sum_{l=0}^{k} p_{\lambda T}(l) m(k-l)$$

Then, we compute the denominator in (3.15) as,

$$Z(S) \;=\; a + \sum_{k=0}^{S-s-1} m(k)G(S-k) \tag{3.18}$$

3.4 Solution Approach

In this section, we attempt to solve the SJRP problem with the policy proposed (mF, s, S). However, given that the optimal solution is rather too complex for SJRP, we propose an efficient heuristic algorithm to obtain near-optimal solution to the problem. Basically, we obtain the time multiple m_i for each item i by solving a deterministic JRP with demand λ_i. Then, we compute the optimal (s_i, S_i) policy by the algorithm of Zheng and Federgruen (1991) for each item i with review period $T_i = m_i F$. This algorithm uses a search on the (s_i, S_i) plane directly. After that we compute the expected total cost, say $TC(F)$ corresponding to the initial value of F, and increment F by 0.01 in either direction. The heuristic stops if the expected total cost $TC(F)$ is not improved for the chosen value of F. Here follows a description of each of the steps of the heuristic.

Step 0: (Initialization step)

Solve the deterministic version of the JRP with demand λ_i for each item i $(i = 1, ..., n)$. Then, set the common cycle for deterministic version to an initial review period F, and the integer time multiples corresponding to the common cycle to m_i^*, for each item i $(i = 1, ..., n)$.

Step 1: (An Iterative step)

For each item $(i = 1, ..., n)$, compute the optimal s_i and S_i corresponding to the current value of the review period $T_i = m_i^* F$, as follows:

(a) Set $S_i^* = y_i^*$, i.e., a value y_i of that minimizes $G_i(y_i)$. Then, repeat $s_i = S_i^* - 1$ until $C_i(s_i, S_i^*) \leq G_i(s_i)$; Set $s_i^* = s$ with the initial best solution so far $C_i^*(T_i) = C_i^*(s_i^*, S_i^*)$.

(b) Set $S_i = S_i^* + 1$. If $G_i(S_i) > C_i^*(T_i)$, then stop as s_i^* and S_i^* are the optimal solution with the cost $C_i^*(T_i)$; else, if $C_i(s_i^*, S_i) < C_i^*(T_i)$; Set $S_i^* = S_i$ and repeat this step; else, proceed to the next step.

(c) Compute the best s corresponding to S_i^*; Set $s_i = s_i^* + 1$ and the new s_i^* is obtained as the smallest value of s_i providing $C_i(s_i, S_i^*) > G_i(s_i + 1)$. Update $C_i^*(T_i) = C_i^*(s_i^*, S_i^*)$ and goto Step b.

and compute the total cost for SJRP, say $TC(F)$, corresponding to the initial value of F.

Step 2: (Improvement step)

Increment F by 0.01, in either direction, then repeat step 1, and update $TC(F)$ and F. Stop if there is no further improvemment in the total cost $TC(F)$ for the current value of F.

Note however that for (mF, S)-2 policy, we use the same heuristics to solve the problem. Therefore, in step 1, we only obtain the optimal S^* instead.

3.5 Performance Evaluation

In this section, we compare the performance of the (mF, s, S) policy and (mF, S)-2 which has a developed heuristic proposed in this chapter to that of the (mF, S)-1, (F, S) which are proposed by Atkins and Iygun (1988), and (F, s, S) policy proposed by Viswanathan (1997) through a numerical results. Note that the (mF, s, S) policy is in general quite robust in that it never performs worse than the policies compared. Though the (mF, s, S) policy is cheaper than the (F, s, S) policy, the cost difference does not appear to be very significant when the inventory holding costs lower than the shortages costs. Therefore, we concentrate on situations where the inventory holding cost is higher than the shortages costs.

For the 12-item problem set, in Table 3.1 gives the problem parameters where ($A = 150$, $h = 30$, $p = 10$, $\pi = 0$) and a for each item are relatively high compared to the joint ordering cost. Also, the table gives the optimal policy solution for the proposed policy, (mF, s, S) policy and compare it to the existing policies. Also, the same data is presented in Table 3.2 with specific-item ordering cost a for each item are relatively low compared to the joint ordering cost and the table gives the optimal policy solution for the (mF, s, S) policy. Note that the integer multiple time m_i for each item i depends on their parameters. Therefore, each item i has an optimal m_i corresponding to a given F.

Moreover, the costs of the proposed policy and the ratio of other policies for different values of parameters p, π and A are given in Table 3.3 and Table 3.4 for moderate specific-item ordering cost a and for high specific-item ordering cost a in example 3.1 and example 3.2 respectively. The numerical results show that the policy proposed (mF, s, S) is the best over all existing policies. The results indicate that the (mF, s, S) policy achieves savings of up to 9% over the (F, s, S) policy for a moderate a and achieves savings of up to 7% over the (F, s, S) policy for a high a. For the non-identical 8-item problem, the Table 3.5 gives the problem parameters for non-identical 8-item problem and the costs of the (mF, s, S) policy and the ratio of other policies. The results reveals that the (mF, s, S) policy achieves savings of up to 1% over the (F, s, S) policy.

The replenishment lead time of the items has a considerable effect on the optimal policy parameters. In Table 3.6, we consider different values of the replenishment lead times L for all items in non-identical 8-item problem as in Table 3.5 and different joint ordering cost A. The results show that the (mF, s, S) policy perform much better when the replenishment lead times for each item i are short. However, for 12-item problem with different values of the replenishment lead times L and different joint ordering cost A is presented in Table 3.7 for both example in Table 3.1 and Table 3.2. The numerical results indicate that the (mF, s, S) policy performs well when the joint ordering cost A is very low compared to the specific-item ordering cost a and when the replenishment lead times are short. In that case, the (mF, s, S) policy achieves savings of up to 10% over the (F, s, S) policy. However, in all cases, the (mF, s, S) policy has the lowest cost, and the policy improves as the replenishment lead times decrease.

Further numerical results are conducted on the 12-item problem for a wide range of parameters. We use the data in Table 3.1 which is for a high specific-item ordering costs, with different values for the parameters A, p, h and $\pi = 0$. The results are presented in Table 3.8. For the problems whose results are presented in Table 3.8, the value $A = \{10, 50, 100, 200, 500\}$, $p = \{10, 50, 100, 200\}$ and $h = \{2, 6, 10\}$. The results show that the (mF, s, S) policy performs as well as (F, s, S) policy when the inventory holding costs are lower than the shortages costs. However, the (mF, S)-2 performs better than the (mF, S)-1 policy. When the inventory holding costs are higher than the shortages costs, the results are presented in Table 3.9, and the data used are in Table 3.1. For the problems parameters, the value $A = \{10, 50, 100, 200, 500\}$, $p = \{2, 6, 10\}$, and $h = \{10, 50, 100, 200\}$. The result reveals that the (mF, s, S) policy performs significantly better than other policies compared, and indicates that the (mF, s, S) policy achieves savings of up to 10% over the (F, s, S) policy. However, the (mF, s, S) policy improves when the joint ordering cost A is low compared to the specific-item ordering cost a in the situation where the inventory holding costs are higher than the shortages costs.

To further verify the robustness of the (mF, s, S) policy, we use the data in Table 3.2 which is for a moderate specific-item ordering costs, with different values for the parameters A, p, h and $\pi = 0$. The results are presented in Table 3.10. For the problems whose results are presented in Table 3.10, the value $A = \{1, 10, 50, 100, 200, 500\}$, $p = \{10, 50, 100, 200\}$ and $h = \{2, 6, 10\}$. The results reveals that the (mF, s, S) policy performs as well as (F, s, S) policy when the inventory holding costs are lower than the shortages costs. However, the (mF, S)-2 performs better than the (mF, S)-1 policy. In the case where the inventory holding costs are higher than the shortages costs, the results are presented in Table 3.11, and the data used are in Table 3.2. For the problems parameters, the value $A = \{1, 10, 50, 100, 200, 500\}$, $p = \{2, 6, 10\}$, and $h = \{10, 50, 100, 200\}$. The result reveals that the (mF, s, S) policy performs significantly better than other policies compared, and indicates that the (mF, s, S) policy achieves savings of up to 10% over the (F, s, S) policy.

Example 3.1 Parameters				(mF, s, S)	(F, s, S)	(mF, S)-2	(mF, S)-1	(F, S)
item	λ	a	L	F^* =1.079	F^* =1.329	F^* =1.079	F^* =0.558	F^* =1.979
1	40	100	0.2	1,3,18	6,21	1,18	1,13	27
2	35	100	0.5	1,11,26	14,29	1,26	1,22	34
3	40	200	0.2	1,0,18	0,21	1,18	1,13	27
4	40	200	0.1	1,0,14	0,17	1,14	1,9	23
5	40	400	0.2	2,0,29	0,21	2,29	1,13	27
6	20	200	1.5	2,24,40	15,37	2,40	1,31	39
7	20	400	1	2,5,30	0,33	2,30	2,25	29
8	20	400	1	2,5,30	0,33	2,30	2,25	29
9	28	600	1	2,6,43	0,47	2,43	2,35	41
10	20	600	1	3,8,36	0,34	3,36	3,28	29
11	20	800	1	3,2,36	0,35	3,36	3,28	29
12	20	800	1	3,2,36	0,35	3,36	3,28	29
Total Cost				4832	4879	4832	6324	5193

Table 3.1: Data as well as optimal policies for 12 items, plus computed cost for (mF, s, S), (F, s, S), (mF, S)-2, (mF, S)-1, and (F, S). Other Parameters, $n = 12$, $A = 150$, $h = 30$, $p = 10$, and $\pi = 0$

Example 3.2 Parameters				(mF, s, S)	(F, s, S)	(mF, S)-2	(mF, S)-1	(F, S)
item	λ	a	L	F^* =0.713	F^* =0.863	F^* =0.733	F^* =0.289	F^* =0.873
1	40	10	0.001	1,0,4	0,5	1,4	1,1	5
2	35	10	0.01	1,0,4	0,5	1,4	1,2	5
3	40	20	0.1	1,0,8	2,9	1,8	1,5	9
4	40	20	0.2	1,4,12	5,13	1,12	1,9	13
5	40	40	0.3	1,3,16	5,17	1,16	1,12	17
6	20	20	0.01	1,0,2	0,3	1,2	1,1	3
7	20	40	0.2	1,0,6	0,6	1,6	1,4	6
8	20	40	0.4	1,0,10	0,10	1,10	1,7	10
9	28	60	0.4	1,0,14	0,15	1,14	1,11	15
10	20	60	0.4	2,2,12	0,10	2,12	2,9	10
11	20	80	0.1	2,0,6	0,4	2,6	2,3	4
12	20	80	0.1	2,0,6	0,4	2,6	2,3	4
Total Cost				1522	1547	1526	2203	1548

Table 3.2: Data as well as optimal policies for 12 items, plus computed cost for (mF, s, S), (F, s, S), (mF, S)-2, (mF, S)-1, and (F, S). Other Parameters, $n = 12$, $A = 150$, $h = 30$, $p = 6$, and $\pi = 0$

Problem Parameters				Cost of Other Policies (Given as a Ratio of the Cost of (mF, s, S))			
p	π	A	(mF, s, S)	(F, s, S)	(mF, S)-2	(mF, S)-1	(F, S)
2	0	20	2320	1.092	1.000	2.099	1.092
		50	2341	1.086	1.000	2.224	1.086
		100	2365	1.080	1.000	2.151	1.080
		150	2389	1.075	1.000	2.243	1.075
		200	2412	1.070	1.000	2.227	1.070
6	2	20	4382	1.080	1.000	1.367	1.081
		50	4416	1.075	1.000	1.427	1.075
		100	4455	1.070	1.000	1.392	1.070
		150	4493	1.066	1.000	1.438	1.066
		200	4531	1.061	1.000	1.430	1.061
2	6	20	4194	1.055	1.000	1.572	1.055
		50	4218	1.051	1.000	1.644	1.051
		100	4242	1.048	1.000	1.610	1.048
		150	4266	1.045	1.000	1.663	1.045
		200	4290	1.042	1.000	1.659	1.042
20	0	20	6012	1.003	1.000	1.109	1.091
		50	6063	1.000	1.000	1.143	1.085
		100	6123	1.000	1.001	1.123	1.081
		150	6181	1.000	1.002	1.151	1.076
		200	6240	1.000	1.003	1.146	1.072
2	10	20	5363	1.049	1.000	1.388	1.049
		50	5393	1.045	1.000	1.445	1.045
		100	5418	1.043	1.000	1.422	1.043
		150	5443	1.040	1.000	1.462	1.040
		200	5468	1.038	1.000	1.462	1.038
10	6	20	6156	1.012	1.000	1.175	1.074
		50	6200	1.009	1.000	1.215	1.069
		100	6249	1.007	1.000	1.194	1.065
		150	6296	1.006	1.000	1.224	1.061
		200	6343	1.004	1.000	1.219	1.057

Table 3.3: Performance of (mF, s, S) over the other policies with different joint ordering cost A and different p, π and $h = 30$ in Example 3.1

Problem Parameters				Cost of Other Policies (Given as a Ratio of the Cost of (mF, s, S))			
p	π	A	(mF, s, S)	(F, s, S)	(mF, S)-2	(mF, S)-1	(F, S)
2	0	20	783	1.068	1.000	2.210	1.068
		50	819	1.049	1.000	2.154	1.049
		100	872	1.028	1.000	2.242	1.028
		150	919	1.015	1.000	2.135	1.015
		200	960	1.007	1.000	2.159	1.007
6	2	20	1862	1.045	1.001	1.333	1.049
		50	1922	1.033	1.000	1.315	1.036
		100	2010	1.020	1.000	1.358	1.022
		150	2083	1.013	1.003	1.323	1.014
		200	2152	1.007	1.001	1.335	1.007
2	6	20	2534	1.040	1.000	1.240	1.040
		50	2594	1.027	1.000	1.237	1.027
		100	2658	1.018	1.000	1.291	1.018
		150	2722	1.009	1.000	1.275	1.009
		200	2769	1.006	1.000	1.301	1.006
20	0	20	2103	1.008	1.005	1.151	1.070
		50	2196	1.005	1.003	1.130	1.051
		100	2330	1.003	1.001	1.158	1.030
		150	2434	1.009	1.010	1.131	1.023
		200	2543	1.005	1.007	1.135	1.012
2	10	20	3262	1.008	1.010	1.106	1.081
		50	3377	1.006	1.009	1.094	1.059
		100	3514	1.006	1.002	1.118	1.040
		150	3621	1.008	1.008	1.104	1.028
		200	3712	1.010	1.004	1.116	1.019
10	6	20	2908	1.006	1.004	1.122	1.060
		50	3006	1.003	1.003	1.110	1.042
		100	3128	1.004	1.001	1.139	1.028
		150	3228	1.007	1.007	1.121	1.020
		200	3326	1.006	1.004	1.129	1.012

Table 3.4: Performance of (mF, s, S) over the other policies with different joint ordering cost A and different p, π and $h = 30$ in Example 3.2

item	Problem Parameters λ	a	L	h	p	π	A	(mF, s, S)	(F, s, S)	(mF, S)-2	(mF, S)-1	(F, S)
1	20	50	0.04	100	50	10						
2	50	50	0.1	100	50	10	50	8844	1.005	1.015	1.070	1.092
3	35	200	0.2	100	30	10	100	9195	1.003	1.009	1.044	1.077
4	90	100	0.1	150	100	70	150	9512	1.002	1.005	1.037	1.067
5	70	50	0.01	150	50	30	200	9804	1.002	1.003	1.051	1.058
6	80	50	0.05	100	70	40	250	10073	1.002	1.010	1.047	1.051
7	50	10	0.4	150	40	10	500	11244	1.000	1.005	1.043	1.030
8	45	10	0.1	100	40	10						

Table 3.5: Performance of (mF, s, S) over the other policies for non-identical 8-items with different joint ordering cost A.

A	L	(mF, s, S)	(F, s, S)	(mF, S)-2	(mF, S)-1	(F, S)
50	0.001	8354	1.023	1.007	1.044	1.095
	0.01	8410	1.020	1.008	1.048	1.094
	0.1	8937	1.006	1.008	1.069	1.081
	0.4	10150	1.000	1.009	1.104	1.064
100	0.001	8709	1.023	1.004	1.026	1.081
	0.01	8757	1.021	1.005	1.028	1.081
	0.1	9259	1.006	1.005	1.045	1.070
	0.4	10425	1.000	1.007	1.076	1.055
150	0.001	9022	1.023	1.004	1.022	1.072
	0.01	9079	1.021	1.003	1.024	1.071
	0.1	9555	1.007	1.003	1.039	1.062
	0.4	10679	1.000	1.005	1.067	1.049
200	0.001	9325	1.022	1.003	1.039	1.063
	0.01	9374	1.020	1.003	1.041	1.063
	0.1	9830	1.007	1.002	1.053	1.055
	0.4	10917	1.000	1.004	1.081	1.043
500	0.001	9702	1.011	1.002	1.025	1.046
	0.01	9751	1.010	1.002	1.026	1.045
	0.1	10136	1.002	1.005	1.045	1.045
	0.4	11145	1.000	1.008	1.076	1.039

Table 3.6: Performance of (mF, s, S) over the other policies for non-identical items with different identical replenishment lead time L for all items and different joint ordering cost A. Other parameters in Table 3.5

A	L	moderate Minor Cost[1]					High Minor Cost[2]				
		(mF,s,S)	(F,s,S)	(mF,S)-2	(mF,S)-1	(F,S)	(mF,s,S)	(F,s,S)	(mF,S)-2	(mF,S)-1	(F,S)
1	0.01	729	1.096	1.000	2.161	1.096	2265	1.102	1.000	2.242	1.102
	0.10	743	1.092	1.000	2.140	1.092	2271	1.102	1.000	2.244	1.102
	0.40	779	1.085	1.000	2.119	1.085	2288	1.100	1.000	2.245	1.100
	0.60	799	1.081	1.000	2.103	1.081	2298	1.099	1.000	2.245	1.099
10	0.01	747	1.079	1.000	2.150	1.079	2276	1.098	1.000	2.157	1.098
	0.10	761	1.077	1.000	2.133	1.077	2282	1.097	1.000	2.159	1.097
	0.40	796	1.070	1.000	2.112	1.070	2299	1.096	1.000	2.161	1.096
	0.60	815	1.067	1.000	2.100	1.067	2309	1.095	1.000	2.162	1.095
50	0.00	799	1.050	1.000	2.165	1.050	2310	1.087	1.000	2.221	1.087
	0.10	812	1.047	1.000	2.151	1.047	2316	1.085	1.000	2.223	1.085
	0.40	846	1.044	1.000	2.134	1.044	2332	1.084	1.000	2.225	1.084
	0.60	864	1.043	1.000	2.124	1.043	2343	1.083	1.000	2.224	1.083
100	0.01	853	1.028	1.000	2.253	1.028	2334	1.081	1.000	2.147	1.081
	0.10	865	1.026	1.000	2.239	1.026	2340	1.080	1.000	2.149	1.080
	0.40	898	1.025	1.000	2.223	1.025	2356	1.079	1.000	2.151	1.079
	0.60	916	1.023	1.000	2.211	1.023	2366	1.078	1.000	2.151	1.078
200	0.01	940	1.008	1.000	2.169	1.008	2382	1.070	1.000	2.223	1.070
	0.10	952	1.007	1.000	2.161	1.007	2387	1.070	1.000	2.225	1.070
	0.40	983	1.006	1.000	2.150	1.006	2403	1.068	1.000	2.227	1.068
	0.60	1000	1.006	1.000	2.142	1.006	2413	1.068	1.000	2.227	1.068
500	0.01	1135	1.000	1.000	2.177	1.000	2498	1.050	1.000	2.164	1.050
	0.10	1145	1.000	1.000	2.176	1.000	2503	1.050	1.000	2.165	1.050
	0.40	1172	1.000	1.000	2.170	1.000	2518	1.049	1.000	2.168	1.049
	0.60	1187	1.000	1.000	2.165	1.000	2528	1.048	1.000	2.168	1.048

[1] Data for moderate minor cost has taken from Example 3.2
[2] Data for high minor cost has taken from Example 3.1.

Table 3.7: Performance of (mF,s,S) over the other policies for identical 12-items with different identical replenishment lead time L for all items and different joint ordering cost A. Other parameters in Table 3.1 and Table 3.2

Problem Parameters				Cost of Other Policies (Given as a Ratio of their Cost to the Cost of (mT, s, S))			
A	p	h	(mF, s, S)	(F, s, S)	(mF, S)-2	(mF, S)-1	(F, S)
10	10	2	2178	1.000	1.012	1.023	1.109
	10	6	3286	1.002	1.000	1.038	1.096
	10	10	3792	1.016	1.000	1.073	1.097
	50	2	2389	1.000	1.051	1.056	1.148
	50	6	3980	1.000	1.045	1.053	1.141
	50	10	4963	1.000	1.041	1.052	1.137
	100	2	2449	1.000	1.066	1.070	1.162
	100	6	4171	1.000	1.064	1.069	1.159
	100	10	5295	1.000	1.061	1.067	1.155
	200	2	2498	1.000	1.079	1.084	1.176
	200	6	4328	1.000	1.081	1.085	1.175
	200	10	5564	1.000	1.079	1.083	1.173
50	10	2	2233	1.000	1.002	1.014	1.086
	10	6	3324	1.000	1.003	1.047	1.088
	10	10	3847	1.005	1.000	1.084	1.085
	50	2	2541	1.000	1.002	1.005	1.083
	50	6	4177	1.000	1.010	1.017	1.092
	50	10	5177	1.000	1.011	1.024	1.094
	100	2	2638	1.000	1.003	1.005	1.083
	100	6	4439	1.000	1.013	1.017	1.093
	100	10	5599	1.000	1.016	1.022	1.096
	200	2	2722	1.000	1.004	1.006	1.083
	200	6	4701	1.000	1.008	1.010	1.086
	200	10	5959	1.000	1.020	1.023	1.099
100	10	2	2258	1.000	1.001	1.007	1.080
	10	6	3363	1.000	1.001	1.034	1.081
	10	10	3886	1.000	1.000	1.068	1.080
	50	2	2566	1.000	1.002	1.002	1.078
	50	6	4237	1.000	1.005	1.007	1.081
	50	10	5236	1.000	1.010	1.016	1.087
	100	2	2663	1.000	1.003	1.003	1.078
	100	6	4511	1.000	1.006	1.007	1.081
	100	10	5663	1.000	1.014	1.016	1.089
	200	2	2748	1.000	1.004	1.004	1.078
	200	6	4745	1.000	1.008	1.008	1.081
	200	10	6026	1.000	1.018	1.018	1.092
200	10	2	2302	1.000	1.001	1.012	1.070
	10	6	3428	1.000	1.002	1.046	1.071
	10	10	3962	1.000	1.000	1.085	1.069
	50	2	2615	1.000	1.002	1.004	1.068
	50	6	4318	1.000	1.004	1.011	1.071
	50	10	5349	1.000	1.007	1.019	1.073
	100	2	2713	1.000	1.003	1.004	1.068
	100	6	4595	1.000	1.006	1.009	1.071
	100	10	5795	1.000	1.009	1.014	1.074
	200	2	2798	1.000	1.004	1.005	1.069
	200	6	4831	1.000	1.007	1.009	1.071
	200	10	6176	1.000	1.011	1.013	1.075
500	10	2	2414	1.001	1.000	1.007	1.049
	10	6	3599	1.006	1.000	1.034	1.049
	10	10	4149	1.000	1.001	1.072	1.050
	50	2	2739	1.000	1.000	1.001	1.047
	50	6	4530	1.000	1.001	1.004	1.049
	50	10	5618	1.000	1.002	1.009	1.050
	100	2	2842	1.000	1.001	1.001	1.047
	100	6	4822	1.000	1.001	1.002	1.047
	100	10	6084	1.000	1.003	1.006	1.050
	200	2	2932	1.000	1.001	1.001	1.047
	200	6	5072	1.000	1.002	1.002	1.047
	200	10	6494	1.000	1.003	1.004	1.048

Table 3.8: Performance of (mF, s, S) Policy over the other policies for 12-item Problem in Table 3.1 when the inventory holding cost is less than the shortages cost

Problem Parameters				Cost of Other Policies (Given as a Ratio of their Cost to the Cost of (mT, s, S))			
A	p	h	(mF, s, S)	(F, s, S)	(mF, S)-2	(mF, S)-1	(F, S)
10	2	10	2159	1.098	1.000	1.453	1.098
	6	10	3263	1.020	1.000	1.138	1.097
	10	10	3792	1.016	1.000	1.073	1.097
	2	50	2351	1.097	1.000	2.679	1.097
	6	50	3957	1.087	1.000	1.723	1.096
	10	50	4963	1.031	1.000	1.460	1.096
	2	100	2397	1.096	1.000	3.646	1.096
	6	100	4124	1.094	1.000	2.239	1.094
	10	100	5263	1.052	1.000	1.836	1.094
	2	200	2433	1.094	1.000	5.017	1.094
	6	200	4252	1.093	1.000	2.986	1.093
	10	200	5492	1.081	1.000	2.390	1.092
50	2	10	2191	1.086	1.000	1.488	1.086
	6	10	3310	1.012	1.000	1.154	1.086
	10	10	3847	1.005	1.000	1.084	1.085
	2	50	2385	1.085	1.000	2.762	1.085
	6	50	4013	1.079	1.000	1.769	1.085
	10	50	5031	1.023	1.000	1.495	1.085
	2	100	2432	1.084	1.000	3.764	1.084
	6	100	4181	1.084	1.000	2.307	1.084
	10	100	5335	1.045	1.000	1.886	1.083
	2	200	2468	1.083	1.000	5.185	1.083
	6	200	4310	1.082	1.000	3.081	1.082
	10	200	5566	1.074	1.000	2.463	1.082
100	2	10	2213	1.080	1.000	1.446	1.080
	6	10	3344	1.010	1.000	1.132	1.080
	10	10	3886	1.000	1.000	1.068	1.080
	2	50	2409	1.080	1.000	2.668	1.080
	6	50	4053	1.078	1.000	1.716	1.080
	10	50	5081	1.021	1.000	1.453	1.080
	2	100	2456	1.079	1.000	3.631	1.079
	6	100	4222	1.078	1.000	2.231	1.078
	10	100	5386	1.044	1.000	1.828	1.078
	2	200	2492	1.078	1.000	5.001	1.078
	6	200	4351	1.077	1.000	2.977	1.077
	10	200	5618	1.073	1.000	2.383	1.077
200	2	10	2257	1.070	1.000	1.489	1.070
	6	10	3410	1.005	1.000	1.155	1.070
	10	10	3962	1.000	1.000	1.085	1.069
	2	50	2457	1.069	1.000	2.766	1.069
	6	50	4131	1.069	1.000	1.771	1.069
	10	50	5178	1.018	1.000	1.496	1.070
	2	100	2503	1.069	1.000	3.772	1.069
	6	100	4302	1.068	1.000	2.310	1.068
	10	100	5487	1.041	1.000	1.889	1.069
	2	200	2540	1.068	1.000	5.196	1.068
	6	200	4432	1.067	1.000	3.088	1.067
	10	200	5721	1.067	1.000	2.468	1.067
500	2	10	2368	1.050	1.000	1.454	1.050
	6	10	3575	1.002	1.000	1.137	1.050
	10	10	4149	1.000	1.001	1.072	1.050
	2	50	2574	1.050	1.000	2.690	1.050
	6	50	4326	1.050	1.000	1.727	1.050
	10	50	5421	1.016	1.000	1.462	1.050
	2	100	2621	1.050	1.000	3.668	1.050
	6	100	4501	1.049	1.000	2.250	1.049
	10	100	5737	1.041	1.000	1.843	1.050
	2	200	2659	1.049	1.000	5.054	1.049
	6	200	4633	1.049	1.000	3.009	1.049
	10	200	5978	1.049	1.000	2.406	1.049

Table 3.9: Performance of (mF, s, S) Policy over the other policies for 12-item Problem in Table 3.1 when the inventory holding cost is greater than the shortages cost

Problem Parameters				Cost of Other Policies (Given as a Ratio of their Cost to the Cost of (mT, s, S))			
A	p	h	(mF, s, S)	(F, s, S)	(mF, S)-2	(mF, S)-1	(F, S)
1	10	2	737	1.000	1.051	1.062	1.147
	10	6	1089	1.000	1.007	1.046	1.101
	10	10	1254	1.015	1.001	1.077	1.094
	50	2	879	1.000	1.129	1.134	1.226
	50	6	1462	1.000	1.118	1.125	1.213
	50	10	1815	1.000	1.105	1.116	1.199
	100	2	929	1.000	1.153	1.159	1.250
	100	6	1596	1.000	1.150	1.156	1.245
	100	10	2028	1.000	1.145	1.151	1.238
	200	2	974	1.000	1.175	1.183	1.272
	200	6	1718	1.000	1.178	1.183	1.271
	200	10	2220	1.000	1.175	1.180	1.267
10	10	2	755	1.000	1.019	1.024	1.095
	10	6	1116	1.000	1.010	1.043	1.086
	10	10	1284	1.007	1.002	1.071	1.078
	50	2	899	1.000	1.035	1.035	1.108
	50	6	1496	1.000	1.075	1.077	1.150
	50	10	1857	1.000	1.073	1.079	1.148
	100	2	950	1.000	1.086	1.087	1.161
	100	6	1632	1.000	1.096	1.096	1.170
	100	10	2072	1.000	1.098	1.099	1.171
	200	2	995	1.000	1.101	1.103	1.175
	200	6	1755	1.000	1.115	1.116	1.188
	200	10	2267	1.000	1.119	1.118	1.190
50	10	2	803	1.000	1.006	1.013	1.053
	10	6	1187	1.000	1.007	1.042	1.055
	10	10	1367	1.001	1.000	1.072	1.049
	50	2	954	1.000	1.012	1.012	1.056
	50	6	1585	1.000	1.022	1.024	1.067
	50	10	1966	1.000	1.030	1.036	1.075
	100	2	1007	1.000	1.015	1.016	1.058
	100	6	1726	1.000	1.034	1.034	1.077
	100	10	2189	1.000	1.039	1.040	1.083
	200	2	1054	1.000	1.025	1.027	1.068
	200	6	1853	1.000	1.040	1.041	1.083
	200	10	2391	1.000	1.046	1.046	1.088
100	10	2	855	1.000	1.004	1.019	1.030
	10	6	1264	1.000	1.006	1.057	1.033
	10	10	1455	1.001	1.001	1.095	1.028
	50	2	1011	1.000	1.007	1.011	1.031
	50	6	1678	1.000	1.015	1.025	1.040
	50	10	2082	1.000	1.019	1.035	1.045
	100	2	1066	1.000	1.009	1.012	1.032
	100	6	1824	1.000	1.019	1.024	1.043
	100	10	2313	1.000	1.024	1.032	1.048
	200	2	1115	1.000	1.011	1.014	1.034
	200	6	1956	1.000	1.022	1.026	1.046
	200	10	2521	1.000	1.028	1.033	1.052
200	10	2	938	1.000	1.001	1.008	1.008
	10	6	1387	1.000	1.003	1.039	1.010
	10	10	1597	1.002	1.002	1.076	1.009
	50	2	1104	1.000	1.003	1.003	1.009
	50	6	1830	1.000	1.005	1.008	1.011
	50	10	2269	1.000	1.008	1.015	1.014
	100	2	1161	1.000	1.004	1.004	1.009
	100	6	1985	1.000	1.006	1.007	1.012
	100	10	2513	1.000	1.009	1.011	1.014
	200	2	1213	1.000	1.005	1.006	1.010
	200	6	2123	1.000	1.008	1.009	1.013
	200	10	2732	1.000	1.011	1.011	1.016
500	10	2	1121	1.000	1.000	1.007	1.000
	10	6	1661	1.000	1.000	1.037	1.000
	10	10	1913	1.000	1.000	1.074	1.000
	50	2	1308	1.000	1.000	1.000	1.000
	50	6	2163	1.000	1.000	1.003	1.000
	50	10	2680	1.000	1.000	1.007	1.000
	100	2	1371	1.000	1.000	1.000	1.000
	100	6	2334	1.000	1.000	1.000	1.000
	100	10	2951	1.000	1.000	1.002	1.000
	200	2	1427	1.000	1.000	1.001	1.000
	200	6	2485	1.000	1.000	1.000	1.000
	200	10	3191	1.000	1.000	1.001	1.000

Table 3.10: Performance of (mF, s, S) Policy over the other policies for 12-item Problem in Table 3.2 when the inventory holding cost is less than the shortages cost

Problem Parameters				Cost of Other Policies (Given as a Ratio of their Cost to the Cost of (mT, s, S))			
A	p	h	(mF, s, S)	(F, s, S)	(mF, S)-2	(mF, S)-1	(F, S)
1	2	10	696	1.097	1.000	1.453	1.097
	6	10	1066	1.043	1.000	1.141	1.095
	10	10	1254	1.015	1.001	1.077	1.094
	2	50	766	1.094	1.000	2.652	1.094
	6	50	1308	1.091	1.000	1.708	1.094
	10	50	1658	1.080	1.000	1.449	1.093
	2	100	789	1.087	1.000	3.559	1.087
	6	100	1377	1.087	1.000	2.184	1.088
	10	100	1775	1.082	1.000	1.794	1.089
	2	200	816	1.076	1.000	4.756	1.076
	6	200	1444	1.076	1.000	2.821	1.077
	10	200	1884	1.074	1.000	2.257	1.078
10	2	10	714	1.079	1.000	1.445	1.079
	6	10	1092	1.033	1.001	1.135	1.078
	10	10	1284	1.007	1.002	1.071	1.078
	2	50	785	1.077	1.000	2.642	1.077
	6	50	1341	1.074	1.001	1.698	1.076
	10	50	1697	1.065	1.002	1.444	1.077
	2	100	806	1.074	1.000	3.566	1.074
	6	100	1407	1.073	1.000	2.183	1.074
	10	100	1811	1.069	1.002	1.796	1.076
	2	200	829	1.067	1.000	4.799	1.067
	6	200	1469	1.067	1.000	2.841	1.067
	10	200	1915	1.066	1.001	2.273	1.069
50	2	10	762	1.050	1.000	1.454	1.050
	6	10	1165	1.024	1.000	1.139	1.050
	10	10	1367	1.002	1.001	1.073	1.050
	2	50	837	1.048	1.000	2.666	1.048
	6	50	1425	1.048	1.000	1.713	1.049
	10	50	1801	1.043	1.001	1.454	1.050
	2	100	858	1.045	1.000	3.612	1.045
	6	100	1495	1.046	1.000	2.210	1.046
	10	100	1922	1.044	1.001	1.811	1.049
	2	200	881	1.040	1.000	4.896	1.040
	6	200	1558	1.040	1.000	2.898	1.041
	10	200	2027	1.041	1.000	2.315	1.043
100	2	10	812	1.028	1.000	1.505	1.028
	6	10	1241	1.020	1.000	1.167	1.028
	10	10	1455	1.001	1.001	1.095	1.028
	2	50	890	1.028	1.000	2.776	1.028
	6	50	1514	1.028	1.000	1.775	1.029
	10	50	1914	1.025	1.000	1.501	1.028
	2	100	912	1.025	1.000	3.750	1.025
	6	100	1585	1.027	1.000	2.289	1.027
	10	100	2038	1.026	1.000	1.872	1.029
	2	200	935	1.021	1.000	5.060	1.021
	6	200	1649	1.022	1.000	2.991	1.022
	10	200	2143	1.024	1.000	2.387	1.025
200	2	10	895	1.007	1.000	1.457	1.007
	6	10	1364	1.008	1.001	1.141	1.008
	10	10	1597	1.002	1.002	1.076	1.009
	2	50	979	1.007	1.000	2.675	1.007
	6	50	1663	1.007	1.001	1.716	1.007
	10	50	2093	1.008	1.003	1.459	1.010
	2	100	1000	1.007	1.000	3.629	1.007
	6	100	1736	1.007	1.000	2.218	1.007
	10	100	2227	1.008	1.002	1.818	1.009
	2	200	1021	1.005	1.000	4.940	1.005
	6	200	1797	1.006	1.000	2.925	1.006
	10	200	2333	1.007	1.001	2.335	1.007
500	2	10	1077	1.000	1.000	1.461	1.000
	6	10	1637	1.000	1.000	1.141	1.000
	10	10	1913	1.000	1.000	1.074	1.000
	2	50	1173	1.000	1.000	2.697	1.000
	6	50	1985	1.000	1.000	1.727	1.000
	10	50	2498	1.000	1.000	1.462	1.000
	2	100	1195	1.000	1.000	3.680	1.000
	6	100	2066	1.000	1.000	2.245	1.000
	10	100	2646	1.000	1.000	1.834	1.000
	2	200	1215	1.000	1.000	5.062	1.000
	6	200	2128	1.000	1.000	2.995	1.000
	10	200	2756	1.000	1.000	2.387	1.000

Table 3.11: Performance of (mF, s, S) Policy over the other policies for 12-item Problem in Table 3.2 when the inventory holding cost is greater than the shortages cost

3.6 Conclusion

In this chapter, we consider a new periodic review policy for solving stochastic joint replenishment problems (SJRPs) referred as to (mF, s, S) policy. The proposed policy assumes a basic review period for all items and the review period of each items is an integer multiple of a basic review period. As well we develop a new and efficient heuristic to (mF, S) policy proposed by Atkins and Iyogun (1988) for solving SJRPs.

Numerical results show that (mF, s, S) outperforms all the policies compared when the inventory holding costs are greater than the shortages costs. Also, we investigate the effects of the non-identical parameters; that are, the inventory holding cost, and the shortages costs on the periodic review policies. However, the data used in the literature have identical parameters such as the inventory holding costs, and the shortages costs. On the other hand, the performance of (mF, s, S) and (mF, S)-2 policy remain on the same level as (F, s, S) policy and even becomes slightly better when the inventory holding costs are lower than the shortages costs. Nevertheless, the magnitude of performance of the proposed policy depends on how to choose the problem parameters.

Chapter 4

Periodic Review Policies

This chapter is concerned with the stochastic joint replenishment problem (SJRP) which is the same problem with its assumptions as in chapter 3. We propose two periodic review policies; namely, (F, Q, S) policy and (F, Q, s, S) policy. We develop expressions for the operating characteristics of both polices and construct the expected total cost function for Poisson demand process. Numerical results have been conducted to study the sensitivity of the policy to various system parameters and to evaluate the performance of the proposed policies over existing policies.

This chapter is organised as follows. In section 4.1 we present an introduction of a new periodic review policy for SJRP and discuss its importance. In section 4.2 we present a mathematical formulation of SJRP using a decomposition approach as well as typical assumptions and notations. A Markov decision process for the single-item inventory model formulation is discussed and a heuristic algorithm to solve SJRP is proposed in Sections 3.3 and 4.4 respectively. In section 4.5 we provide comparative analysis and numerical results for the periodic review policies to help evaluate their performance. Finally, a conclusion is presented in section 4.6.

4.1 Introduction

We consider a stochastic joint replenishment problem (SJRP) where demand that cannot be satisfied immediately is backordered, so as to minimise the expected total ordering, inventory holding and shortages costs per unit time. We propose two new periodic review policies for solving stochastic joint replenishment problems (SJRPs) referred as to (F, Q, S) policy and (F, Q, s, S) policy. The proposed policies base the replenishment decisions on the aggregate demand that have accumulated for all items during a replenishment order cycle. In the (F, Q, S) policy, the inventory is reviewed periodically every F periods, and all items are ordered up to their levels S only if the aggregate demand during a replenishment order cycle reaches Q whereas in (F, Q, s, S) policy, the inventory is reviewed periodically every F periods, and all items with inventory positions below their s are ordered up to their levels S only if the aggregate demand during a replenishment order cycle reaches Q. These policies combine features of both periodic and continuous review policies into an effective policy. (F, Q, S) policy combines features of (F, S) policy proposed Atkins and Iyogun (1988) and (Q, S) policy proposed by Renberg and Planche (1967). (F, Q, s, S) policy combines features of (F, s, S) policy proposed by Viswanathan (1997) policy and (Q, s, S) policy by Nielsen and Larsen (2005). Consequently, the (F, s, S) policy is a special case of the (F, Q, s, S) policy when $Q = 1$, and the (Q, s, S) policy is a special case of the (F, Q, s, S) policy when $F \to 0$. Through this chapter, we use the notation and the typical assumption used in previous chapter. Also, we define Q as an aggregate demand for all items during a repelinshment order cycle.

4.2 Period Review Policy (F, Q, S)

In (F, Q, S) policy, an order need not be placed at each review period F. Hence, the time between the placing of two successive repelinshment orders (i.e., the length of a repelinshment order cycle) will always be an integral multiple of the time F between review periods. The aggregate demand for all items follow a Poisson process with parameter μ, where $\mu = \sum_i \lambda_i$. We assume that the cumulative aggregate demand X_j for all items until j^{th} period, where $j = 1, 2, ...$ The demands in periods are independent and identically distributed (iid). Let $p_{\mu F}(x)$ be the probability that x units will be demanded in one period. Then, since the demands in different periods are assumed to be independent, $p_{\mu jF}(x)$ is the probability that x units are demanded in j periods. Let $\hat{N}(Q)$ be a random variable that represents the first period in which the cumulative aggregate demand exceeds Q.

$$\hat{N}(Q) = \arg\min \{j | X_j \geq Q\} \tag{4.1}$$

Let,

$\hat{m}(x)$ = the expected number of periods when the aggregate demand exceeds x during a replenishment order cycle, where $x = 0, 1, ..., Q - 1$

$\hat{M}(Q)$ = the expected total number of periods when the aggregate demand exceeds Q during a replenishment order cycle,

We can compute $\hat{m}(x)$, and $\hat{M}(Q)$ as the same way we compute (3.15) and (3.17) in chapter 3.

$$\hat{m}(x) = \begin{cases} \frac{1}{1-p_{\mu F}(0)}, & x = 0; \\ \sum_{k=0}^{x} p_{\mu F}(k)\hat{m}(x-k), & x = 1, 2, ..., Q - 1. \end{cases} \tag{4.2}$$

$$\hat{M}(Q) = \sum_{x=0}^{Q-1} \hat{m}(x) \tag{4.3}$$

4.2.1 Model Formulation

First, we decompose the problem into n single-item inventory problems. For a given review period F, and a given aggregate demand Q, the optimal S_i policy for each item i can be computed using the second part of the algorithm of Zheng and Federgruen (1991) with a review period F and an aggregate demand Q. The expected total cost (TC(F,Q)) (F, Q, S) policy corresponding to F and Q can then be obtained as

$$TC(F,Q) \;=\; \frac{A}{F\hat{M}(Q)} + \sum_i C_i^*(F,Q) \tag{4.4}$$

where $C_i^*(F,Q)$ is the cost per unit time of the optimal S_i policy for item i corresponding to a review period of F and Q, and item i only bears the item-specific ordering cost.

4.2.2 Single-item Model Formulation

In this section, we consider the case of a single item problem and to simplify the notation let us drop the product index i. We derive the cost expression for (F,Q,S) policy in the case where demands are generated by Poisson processes and units are demanded one a time.

Single-item Cost Function

Under an (F,Q,S) policy, an item inventory position follows a regenerative process, and hence has a steady state distribution. Each time placing an order and a new replenishment order cycle begins, the item inventory position is brought up to level S and we have a regeneration. Note that we place an order for a specific item when the aggregate demand for all items exceeds Q and the item's inventory position just below S. That means we place an order for that item if there is at least one demand during a replenishment order cycle. We assume that the cumulative aggregate demand X_j for all items until j^{th} period, and the cumulative demand Y_j for a specific item until j^{th} period, where $j = 1, 2, ...$ The demands X_j and Y_j in periods are independent and identically distributed (iid) respectively. Let $p(k|x)$ be the probability that k units for a specific item included in the aggregate demand x during a replenishment order cycle. Also, let $\bar{N}(1,Q)$ be a random variable that represents the first period in which the cumulative demand for specific item exceeds 1 and the cumulative aggregate demand for all items exceeds Q.

$$\bar{N}(1,Q) = \arg\min \{j | Y_j \geq 1, X_j \geq Q\} \tag{4.5}$$

Let,

$G(y)$	$=$	the expected cost of inventory holding and shortages function for a period of length T, where an inventory position is y at the beggining of the period; y integer,
$Z(S)$	$=$	the expected cost when the inventory position $IP = S$ during a replenishement order cycle,
$\bar{m}(k,x)$	$=$	the expected number of periods when the aggregate demand for all items exceed x and the cumulative demand for a specific item exceeds k during a replenishment order cycle, where $k = 0, 1, ..., S - s - 1$,
$\bar{M}(1,Q)$	$=$	the expected total number of periods when the aggregate demand for all items exceeds Q and the cumulative demand for a single item exceeds 1 during a replenishment order cycle,

Hence, by the Proposition (5.9) of the regenerative processes in Ross (1970), we have that

$$C(S) = \frac{Z(S)}{F\bar{M}(1,Q)} \tag{4.6}$$

We need to obtain the expressions for the denominator and the numerator in (4.6). Define $Z_j = X_j - Y_j$, we compute the numerator $\bar{M}(1,Q)$ as follows

$$
\begin{aligned}
P(\bar{N}(1,Q) > j) &= 1 - P(Y_j \geq 1, X_j \geq Q) \\
&= 1 - \sum_{k=1}^{\infty} P(Y_j = k) P(Z_j \geq Q - k) \\
&= 1 - \sum_{k=1}^{\infty} P(Y_j = k) \Big(1 - P(Z_j < Q - k)\Big) \\
&= 1 - \sum_{k=1}^{\infty} P(Y_j = k) + \sum_{k=1}^{\infty} P(Y_j = k) P(Z_j < Q - k) \\
&= P(Y_j = 0) + \sum_{k=1}^{\min\{S-1,Q-1\}} \sum_{x=k}^{Q-1} P(Y_j = k) P(Z_j = x - k) \\
&= P(Y_j = 0) + \sum_{k=1}^{\min\{S-1,Q-1\}} \sum_{x=k}^{Q-1} p_{\lambda j F}(k) p_{(\mu-\lambda) j F}(x - k) \\
&= P(Y_j = 0) + \sum_{k=1}^{\min\{S-1,Q-1\}} \sum_{x=k}^{Q-1} \frac{(\lambda j F)^k e^{-\lambda j F}}{k!} \frac{((\mu - \lambda) j F)^{x-k} e^{-(\mu-\lambda) j F}}{(x - k)!} \\
&= P(Y_j = 0) + \sum_{k=1}^{\min\{S-1,Q-1\}} \sum_{x=k}^{Q-1} \frac{(\mu j F)^x e^{-\mu j F}}{x!} p(k|x)
\end{aligned}
$$

Then, we compute the expected value of $\bar{N}(1,Q)$ as

$$
\begin{aligned}
\bar{M}(1,Q) &= I\!E(\bar{N}(1,Q)) \\
&= \sum_{j=0}^{\infty} P(\bar{N}(1,Q) > j) \\
&= \sum_{j=0}^{\infty} \Big[P(Y_j = 0) + \sum_{k=1}^{\min\{S-1,Q-1\}} \sum_{x=k}^{Q-1} \frac{(\mu j F)^x e^{-\mu j F}}{x!} p(k|x)\Big] \\
&= \frac{1}{1 - e^{-\lambda F}} + \sum_{k=1}^{\min\{S-1,Q-1\}} \sum_{x=k}^{Q-1} \hat{m}(x) p(k|x) \\
&= \frac{1}{1 - e^{-\lambda F}} + \sum_{k=1}^{\min\{S-1,Q-1\}} \sum_{x=k}^{Q-1} \bar{m}(k, x)
\end{aligned}
$$

where $p(k|x)$ is the probability that k units demended for a specific item included in the aggregate demand for all items x. That is a binomial distribution (see

Appendix B). Then, we compute the denominator in (4.6) as

$$Z(S) \;=\; a + \frac{G(S)}{1-e^{-\lambda F}} + \sum_{k=1}^{\min\{S-1,Q-1\}} \sum_{x=k}^{Q-1} \bar{m}(k,x)G(S-k) \tag{4.7}$$

where $G(\cdot)$ is the equation (3.13) computed in chapter 3.

4.3 Periodic Review Policy (F, Q, s, S)

We apply the same methodology as in (F, Q, S) policy in the previous section.

4.3.1 Model Formulation

Assume that a joint ordering cost A is incurred each time the aggregate demand exceeds Q. For a given review period F, and a given aggregate demand Q, the optimal $s_i, S_i)$ policy for each item i can be computed using the algorithm of Zheng and Federgruen (1991). Therefore, The expected total cost (TC(F,Q)) for (F, Q, s, S) policy corresponding to F and Q can then be obtained approximately as

$$TC(F,Q) \;=\; \frac{A}{FM(Q)} + \sum_i C_i^*(F,Q) \tag{4.8}$$

where $C_i^*(F,Q)$ is the cost per unit time of the optimal (s_i, S_i) policy for item i corresponding to a review period of F and Q, and item i only bears the item-specific ordering cost. The reason why it is only approximate cost is due to the possibility of none of the items i is with an inventory position drops or below their s_i at the time of the replenishment possibility.

4.3.2 Single-item Model Formulation

In this section, we consider the case of a single item problem and to simplify the notation let us drop the product index i. We derive the cost expression for (F, Q, s, S) policy in the case where demands are generated by Poisson processes and units are demanded one a time.

Single-item Cost Function

We apply the same methodology as in (F, Q, S) policy. Let us define the random variable $\ddot{N}(S-s, Q)$ that represents the first period in which the cumulative demand for specific item exceeds $S-s$ when the cumulative aggregate demand for all items exceeds Q.

$$\ddot{N}(S-s, Q) = \arg\min \{j | Y_j \geq S-s, X_j \geq Q\} \tag{4.9}$$

Let,

$Z(S)$	$=$	the expected cost when the inventory position $IP = S$ during a replenishement order cycle,
$m(k)$	$=$	the expected number of periods when the cumulative demand for a specific item exceeds k during a replenishment order cycle, where $x = 0, 1, ..., S-s-1$,
$\ddot{M}(S-s, Q)$	$=$	the expected total number of periods when the aggregate demand for all items exceeds Q and the cumulative demand for a specific item exceeds $S-s$ during a replenishment order cycle,

Hence, by the Proposition (5.9) of the regenerative processes in Ross (1970), we have that

$$C(s, S) = \frac{Z(S)}{F\ddot{M}(S-s, Q)} \tag{4.10}$$

We need to obtain the expressions for the denominator and the numerator in (4.6). First, we compute the numerator $\ddot{M}(S-s, Q)$ as follows

$$
\begin{aligned}
P(\ddot{N}(S-s,Q)>j) &= 1-P(Y_j\geq S-s, X_j\geq Q)\\
&= 1-\sum_{k=S-s}^{\infty}P(Y_j=k)P(Z_j\geq Q-k)\\
&= 1-\sum_{k=S-s}^{\infty}P(Y_j=k)\Big(1-P(Z_j<Q-x)\Big)\\
&= 1-\sum_{k=S-s}^{\infty}P(Y_j=k)+\sum_{k=S-s}^{\infty}P(Y_j=k)P(Z_j<Q-k)\\
&= \sum_{k=0}^{S-s-1}P(Y_j=k)+\sum_{k=S-s}^{\min\{S-1,Q-1\}}\sum_{x=k}^{Q-1}P(Y_j=k)P(Z_j=x-k)\\
&= \sum_{k=0}^{S-s-1}P(Y_j=k)+\sum_{k=S-s}^{\min\{S-1,Q-1\}}\sum_{x=k}^{Q-1}p_{\lambda jF}(k)p_{(\mu-\lambda)jF}(x-k)\\
&= \sum_{k=0}^{S-s-1}P(Y_j=k)+\sum_{k=S-s}^{\min\{S-1,Q-1\}}\sum_{x=k}^{Q-1}\frac{(\lambda jF)^k e^{-\lambda jF}}{k!}\frac{((\mu-\lambda)jF)^{x-k}e^{-(\mu-\lambda)jF}}{(x-k)!}\\
&= \sum_{k=0}^{S-s-1}P(Y_j=k)+\sum_{k=S-s}^{\min\{S-1,Q-1\}}\sum_{x=k}^{Q-1}\frac{(\mu jF)^x e^{-\mu jF}}{x!}p(k|x)
\end{aligned}
$$

Then, we compute the expected value of $\ddot{N}(S-s,Q)$ as

$$
\begin{aligned}
\ddot{M}(S-s,Q) &= \mathbb{E}(\ddot{N}(S-s,Q))\\
&= \sum_{j=0}^{\infty}P(N>j)\\
&= \sum_{j=0}^{\infty}\Big[\sum_{k=0}^{S-s-1}P(Y_j=k)+\sum_{k=1}^{\min\{S-1,Q-1\}}\sum_{x=k}^{Q-1}\frac{(\mu jF)^x e^{-\mu jF}}{x!}p(k|x)\Big]\\
&= \sum_{k=0}^{S-s-1}m(k)+\sum_{k=S-s}^{\min\{S-1,Q-1\}}\sum_{x=k}^{Q-1}\hat{m}(x)p(k|x)\\
&= \sum_{k=0}^{S-s-1}m(k)+\sum_{k=S-s}^{\min\{S-1,Q-1\}}\sum_{x=k}^{Q-1}\bar{m}(k,x)
\end{aligned}
$$

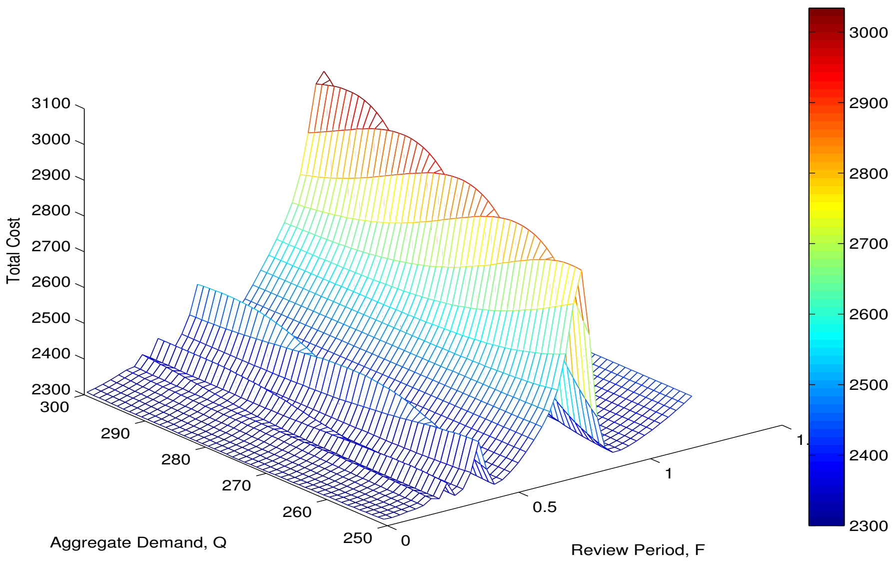

Figure 4.1: The Shape of the Expected Total Cost for (F, Q, S) Policy. Note: The data used is the 12-item problem with parameters as in Table 4.1

where $m(k)$ for a specific item is

$$m(k) = \begin{cases} \frac{1}{1-p_{\lambda T}(0)}, & k = 0; \\ \sum_{l=0}^{k} p_{\lambda F}(k) m(k-l), & k = 1, 2, ..., S-s-1. \end{cases} \tag{4.11}$$

and $p(k|x)$ is the probability that k units demended for a specific item included in the aggregate demand for all items x. That is a binomial distribution (see Appendix B). Then, we compute the denominator in (4.10) as

$$Z(S) = a + \sum_{k=0}^{S-s-1} m(k)G(S-k) + \sum_{k=S-s}^{Q-1} \sum_{x=k}^{Q-1} \bar{m}(k,x)G(S-k) \tag{4.12}$$

where $G(\cdot)$ is the equation (3.13) computed in chapter 3.

4.4 Solution Procedure

In this section, we use a two-dimensional search approach to obtain the near-optimal solution for the problem using two policies proposed in previous section within the given search space on the value of F and the value of Q. Operations

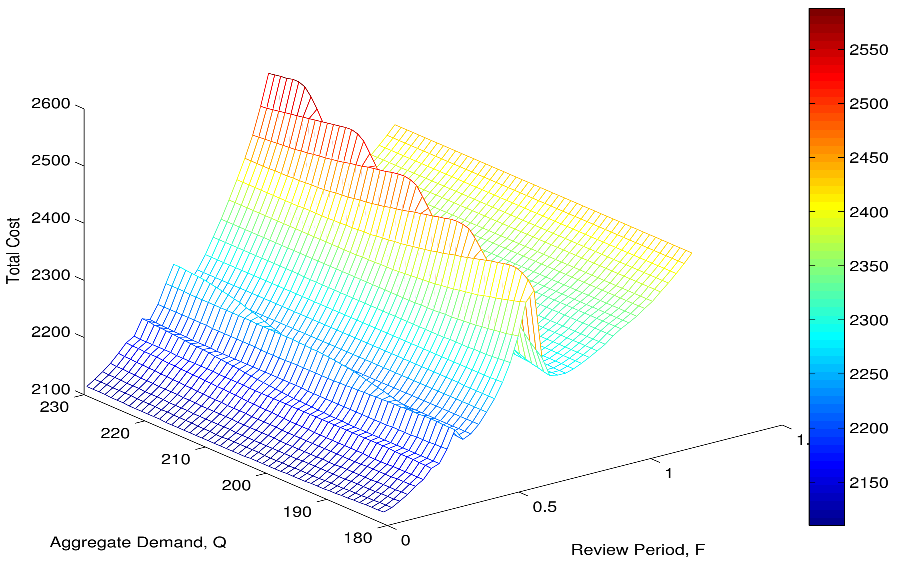

Figure 4.2: The Shape of the Expected Total Cost for (F, Q, s, S) Policy. Note: The data used is the 12-item problem with parameters as in Table 4.1

managers perform search on all feasible values of F and Q within space determined until the optimal $TC(F, Q)$ is found. However, if the endpoints gives better results than any value within the range determined for F or Q, we extend the range for F or Q respectively. The shape of the expected total cost $TC(F, Q)$ for (F, Q, S) policy, and (F, Q, S) policy respectively are illustrated in Figure (4.1) and Figure (4.2). As can be seen in Figure (4.1) and Figure (4.2) it is quite obvious when one goes far away from optimum.

Basically, the heuristic enumerates all feasible values of F and Q within the space determined. For a given F, and Q, we compute the optimal (s_i, S_i) by the algorithm of Zheng and Federgruen (1991) for each item i with review period $T_i = m_i F$. The algorithm search moves vertically up and horizontally right to update S_i and s_i during each step, and eventually reaches the optimal pair (s^*, S^*). Then, we compute the expected total cost, say $TC(F)$ corresponding to all values of F and Q. Amongst the solutions identified for $TC(F)$ along with the one corresponding to F and the one corresponding Q, choose the solution with minimum expected total cost as the best solution. However, this heuristic produces the near-optimal solution of the problem. This heuristic is composed of the following three steps.

Step 0: (Initialization step)

The search space consists of $F \in [0.01, 0.5]$ and $Q \in [1, 200]$ with increments $\Delta_F = 0.01$ and $\Delta_Q = 1$ respectively. If the endpoints of the intervals for F or Q give better results than others, we extend the interval, otherwise we narrow it.

Step 1: (An Iterative step)

For a given review period F, and a given aggregate consumption Q, compute the optimal s_i, S_i and the corresponding cost $C_i(F, Q)$ for each item $(i = 1, ..., n)$, as follow:

(a) Set $S_i^* = y_i^*$, i.e., a value y_i of that minimizes $G_i(y_i)$. Then, Repeat $s_i = S_i^* - 1$ until $C_i(s_i, S_i^*) \leq G_i(s_i)$; Set $s_i^* = s$ with the initial best solution so far $C_i^*(T_i) = C_i^*(s_i^*, S_i^*)$.

(b) Set $S_i = S_i^* + 1$. If $G_i(S_i) > C_i^*(T_i)$, then stop as s_i^* and S_i^* are the optimal solution with the cost $C_i^*(T_i)$; else, if $C_i(s_i^*, S_i) < C_i^*(T_i)$; Set $S_i^* = S_i$ and repeat this step; else, proceed to the next step.

(c) Compute the best s corresponding to S_i^*; Set $s_i = s_i^* + 1$ and the new s_i^* is obtained as the smallest value of s_i providing $C_i(s_i, S_i^*) > G_i(s_i + 1)$. Update $C_i^*(T_i) = C_i^*(s_i^*, S_i^*)$ and goto Step b.

and compute the total cost for SJRP, say $TC(F, Q)$, corresponding to the value of F and Q.

*Step 2***:**

Compute the corresponding total cost, say $TC(F, Q)$, over all values of F and Q. Amongst the solutions identified along with the one corresponding to F and Q, choose the solution with minimum total cost as the best solution.

Note however that for (F, Q, S) policy, we use the same heuristics to solve the problem. Therefore, in step 1, we only obtain the optimal S^* instead.

4.5 Comparative Analysis and Numerical Results

In this section, we compare the performance of the (F, Q, s, S) policy and (F, Q, S) policy to other existing policies through a numerical results. The existing policies are (F, S), (mF, S), (F, s, S), (Q, S), (Q, s, S), and the can-order policies (s, c, S), and (s, c, S)-C as computed in Federgruen et. al. (1984) and Melchiors (2002) respectively. Also, we obtain the lower bound which suggested in Atkin and Iyogun (1988). Therefore, we use the 12-item problems and the 8-item problems presented in Atkins and Iyogun (1988). We use the ratio of the existing policy cost to the (F, Q, s, S) policy cost.

For the 12-item problem, in Table 4.1 gives the problem parameters where ($A = 150$, $h = 6$, $p = 0$, and $\pi = 30$). Also, the table gives the optimal policy solution for the proposed policies, that are, (F, Q, s, S) policy and (F, Q, S) policy and compare them with the existing policies. Moreover, the costs of the proposed policies and the ratio of other policies for different values of parameters are given in Table 4.2. For the 8-item problem, the Table 4.3 gives the problem parameters for identical items. It is interesting to note that the solution to the (F, Q, s, S) policy is significantly better than existing policies.

The numerical results indicate that the first policy proposed (F, Q, s, S) is the best over all existing policies. In the last part of the Table 4.2, the inventory holding costs are relatively high compared to the shortages costs, the (F, Q, S) policy performs significantly better than (Q, s, S) as the joint ordering cost A increases. The last part of the Table 4.2 indicates that the (F, Q, S) policy achieves savings of at least 4% over the (Q, s, S) policy. In Table 4.4, we have six examples where 12 items are identical with their costs and demand. The results obtained show that the (F, Q, s, S) policy dominates over all polices whereas the (F, Q, S) policy performs as well as the (Q, s, S) policy.

Further numerical results are conducted on the 12-item problem for a wide range of parameters. We use the same data in Table 4.1, with different values for the parameters A, p, h and $\pi = 0$ which generated from Viswanthan (1997). The results are presented in Table 4.5 and Table 4.6. For the problems whose results are presented in Table 4.5, the value $A = \{20, 50, 100, 200, 500\}$, $p = \{10, 50, 100, 200\}$ and $h = \{2, 6, 10\}$. The results show that the (F, Q, s, S) policy dominates all other existing polices. The (F, Q, S) policy performs better than the other policies for most of the problems. The (Q, s, S) policy performs better than the (F, Q, S) policy when the joint ordering cost A is low and the shortages costs are high compared to the inventory holding cost. For example, $A = 20$, $p = 200$, and $h = 10$. The (F, Q, S) policy performs well when the joint ordering cost A is high. For example, when $A = 500$, $p = 10$, and $h = 2$.

To further verify the robustness of the proposed policies, the values of p and h are increased by a factor of 100. This corresponds to an increase in demand rate Viswanthan (1997). The corresponding results are presented in Table 4.6. For these problems, the (F, Q, s, S) policy outperforms the existing policies whereas (F, Q, S) policy perform worse when the inventory holding costs and shortages costs are very high.

The replenishment lead times of the items and the specific item demand rate also have a considerable effects on the optimal policy parameters. Therefore, we investigate the effects of these factors using the data from ´Ozkaya et. al. (2006). The results are presented in Tables 4.7, 4.8 and 4.9.

In Table 4.7, we consider 8-items with identical parameters where $A = 150$, $h = 6$, $p = 0$, $\pi = 30$, $L = 0.2$ and different $a = \{0, 20, 40, 60\}$ and identical demand rates, $\lambda = \{20, 40, 60, 80\}$. In all cases, the (F, Q, s, S) policy dominates the existing policies. The results indicate that the (F, Q, s, S) policy achieves savings of up to 10% over the (F, s, S) policy and 9% over the (Q, s, S) policy. In Table 4.8, we consider n-items with identical parameters where $A = 150$, $\mu = \sum_i \lambda = 320$, $a = 20$ $h = 6$, $p = 0$, $\pi = 30$, and $L = \{0, 2, 0.4, 0.6\}$ and the number of items, $n = \{2, 4, 6, 8, 10, 12\}$. Note that individual demand rates are equal to each other. The results reveals that the (F, Q, s, S) policy achieves savings of at least 2% over the (F, s, S) policy and 2% over the (Q, s, S) policy. The (F, Q, S) policy performs significantly better than (Q, s, S) when the joint ordering cost A is high. In Table 4.9, we consider a 4-item problem with identical parameters where $A = 150$, $\mu = \sum_i \lambda_i = 320$, $a = 20$ $h = 6$, $p = 0$, $\pi = 30$, and $L = 0.2$. Note that various groupings of demand rates among the items are considered. The results reveals that the (F, Q, s, S) policy achieves savings of at least 7% over the (F, s, S) policy and 4% over the (Q, s, S) policy. However, The (F, Q, S) policy performs as well as (Q, s, S) in all cases.

Problem Parameters				(F,Q,s,S) $T=0.01$ $Q=209$	(F,Q,S) $T=0.01$ $Q=275$	(F,S) $T=0.8$	(mF,S) $T=0.65$	(F,s,S) $T=0.557$	(Q,S) $Q=275$	(Q,s,S) $Q=195$	(s,c,S)	(s,c,S)-C	LB
item	a	λ	L										
1	10	40	0.2	12,40	46	46	1,41	33,37	46	33,37	8,34,46	9,26,38	11,39,065
2	10	35	0.5	23,47	52	52	1,47	40,45	52	40,45	17,43,54	19,36,47	22,47,065
3	20	40	0.2	12,41	46	46	1,41	31,37	46	31,37	8,34,49	9,24,41	11,39,065
4	20	40	0.1	7,36	41	42	1,36	27,33	42	27,33	4,27,44	5,19,36	6,34,065
5	40	40	0.2	11,42	46	46	1,41	29,37	46	30,37	8,29,53	9,22,46	11,39,065
6	20	20	1.5	36,52	53	53	1,50	43,50	53	43,50	23,46,58	28,42,55	35,52,065
7	40	20	1.0	24,43	42	42	1,40	31,46	42	31,47	14,33,50	18,30,47	24,43,0.82
8	40	20	1.0	24,43	42	42	1,40	31,46	42	31,47	14,33,50	18,30,47	24,43,0.82
9	60	28	1.0	33,60	58	58	1,54	42,65	57	43,65	25,44,69	27,40,64	33,60,0.85
10	60	20	1.0	23,46	42	42	2,50	30,48	42	30,48	13,32,53	18,29,50	23,46,1
11	80	20	1.0	23,49	42	42	2,50	29,50	42	29,49	12,31,55	17,28,52	23,49,1.15
12	80	20	1.0	23,49	42	42	2,50	29,50	42	29,49	12,31,55	17,28,52	23,49,1.15
Total Cost				2110	2300	2322	2291	2267	2304	2252	2620	2268	2047

Table 4.1: Data as well as optimal policies for 12 items, plus computed cost for (F,Q,s,S), (F,Q,S) and other policies in Literature. Other Parameters, $n=12$, $A=150$, $h=6$, $p=30$, and $\pi=0$

Problem Parameters				(F,Q,s,S)	Cost of Other Policies (Given as a Ratio of the Cost of (F,Q,s,S))							
h	p	π	A		(F,Q,S)	(mT,S)	(F,s,S)	(Q,S)	(Q,s,S)	(s,c,S)	(s,c,S)-C	LB
6	0	30	150	2110	1.090	1.086	1.074	1.092	1.067	1.242	1.075	0.970
2	0	30	50	1019	1.149	1.113	1.101	1.152	1.093	1.152	1.089	0.981
			100	1078	1.122	1.111	1.099	1.124	1.091	1.231	1.123	0.975
			150	1130	1.105	1.100	1.098	1.107	1.089	1.291	1.141	0.970
			200	1176	1.093	1.096	1.093	1.096	1.085	1.337	1.153	0.967
			250	1218	1.084	1.092	1.090	1.086	1.081	1.335	1.158	0.964
2	30	0	20	822	1.116	1.102	1.078	1.160	1.073	1.121	1.061	0.991
			50	861	1.088	1.097	1.085	1.136	1.079	1.294	1.097	0.989
			100	912	1.060	1.102	1.088	1.114	1.081	1.363	1.134	0.988
			150	954	1.044	1.097	1.093	1.103	1.086	1.415	1.159	0.991
			200	986	1.041	1.104	1.101	1.103	1.094	1.464	1.176	0.995
6	30	0	100	1508	1.041	1.112	1.076	1.110	1.071	1.258	1.100	0.990
			150	1562	1.034	1.109	1.092	1.109	1.086	1.315	1.118	1.001
			200	1611	1.029	1.119	1.104	1.109	1.098	1.361	1.131	1.010
20	30	0	20	2239	1.056	1.145	1.038	1.129	1.036	1.064	1.034	0.994
			50	2310	1.041	1.137	1.051	1.117	1.047	1.132	1.054	1.000
			100	2398	1.031	1.172	1.068	1.113	1.064	1.221	1.072	1.016
			150	2475	1.025	1.160	1.084	1.113	1.080	1.294	1.092	1.028
			200	2546	1.020	1.173	1.097	1.113	1.093	1.351	1.108	1.040

Table 4.2: Performance of (F,Q,s,S), and (F,Q,S) over other policies for 12-item problem in Table 4.1 with different joint ordering cost A and different parameters of h, p, and π

Parameters			(F,Q,s,S)	Cost of Other Policies (Given as a Ratio of the Cost of (F,Q,s,S))							
Problem				(F,Q,S)	(mF,S)	(F,s,S)	(Q,S)	(Q,s,S)	(s,c,S)	(s,c,S)-C	LB
1	$\lambda = 40$	$L = 0.2$	1437	1.069	1.085	1.085	1.069	1.070	1.342	1.139	0.944
2	$\lambda = 40$	$L = 0.4$	1521	1.049	1.062	1.062	1.049	1.048	1.309	1.106	0.953
3	$\lambda = 40$	$L = 0.6$	1587	1.038	1.049	1.049	1.038	1.038	1.288	1.131	0.960
4 $\lambda_{1-4} = 20$	$\lambda_{5-8} = 60$	$L = 0.2$	1416	1.072	1.091	1.089	1.073	1.072	1.320	1.115	0.952
5 $n = 4$	$\lambda = 80$	$L = 0.2$	1188	1.040	1.064	1.064	1.039	1.039	1.257	1.039	0.949

Table 4.3: Performance of (F,Q,s,S), and (F,Q,S) over other policies for identical item problem. Other Parameters, $n = 8$, $A = 150$, $a = 20$, $h = 6$, $p = 0$, and $\pi = 30$

A	a	(F, Q, s, S)	Cost of Other Policies (Given as a Ratio of the Cost of (F, Q, s, S))					
			(F, Q, S)	(F, s, S)	(Q, s, S)	(s, c, S)	(s, c, S)-C	LB
500	10	1357	1.027	1.035	1.027	1.426	1.052	0.936
500	50	1613	1.041	1.048	1.042	1.286	1.085	0.955
150	10	1051	1.039	1.048	1.039	1.138	1.052	0.951
150	50	1381	1.059	1.066	1.059	1.118	1.096	0.973
50	10	929	1.054	1.059	1.052	1.064	1.062	0.965
50	50	1301	1.075	1.082	1.075	1.067	1.059	0.987

Table 4.4: Performance of (F, Q, s, S), and (F, Q, S) over other policies for identical 12-item problem. Other Parameters, $n = 12$, $\lambda = 10$, $L = 1$, $h = 6$, $\pi = 30$, and $p = 10$ for the two first examples, and $p = 0$ in the remaining four.

Problem Parameters				(F,Q,s,S)	Cost of Other Policies (Given as a Ratio of the Cost of (F,Q,s,S))						
A	π	p	h		(F,Q,S)	(mF,S)	(F,s,S)	(Q,S)	(Q,s,S)	(s,c,S)	LB
20	0	10	2	736	1.056	1.086	1.057	1.136	1.054	1.094	0.997
		10	6	1125	1.035	1.110	1.036	1.119	1.034	1.076	0.999
		10	10	1322	1.030	1.148	1.026	1.110	1.024	1.067	1.002
		50	2	855	1.144	1.116	1.087	1.171	1.080	1.088	0.989
		50	6	1482	1.119	1.109	1.065	1.155	1.059	1.074	0.991
		50	10	1888	1.103	1.110	1.055	1.146	1.050	1.069	0.990
		100	2	894	1.169	1.131	1.097	1.185	1.086	1.087	0.989
		100	6	1608	1.147	1.118	1.067	1.167	1.064	1.072	0.989
		100	10	2102	1.135	1.114	1.057	1.156	1.054	1.064	0.988
		200	2	930	1.186	1.143	1.105	1.196	1.090	1.085	0.988
		200	6	1723	1.165	1.128	1.070	1.175	1.067	1.067	0.987
		200	10	2297	1.154	1.121	1.060	1.164	1.057	1.060	0.988
50	0	10	2	762	1.043	1.089	1.075	1.126	1.070	1.192	1.009
		10	6	1161	1.022	1.109	1.053	1.111	1.050	1.165	1.014
		10	10	1362	1.018	1.146	1.043	1.104	1.021	1.148	1.021
		50	2	895	1.117	1.107	1.094	1.145	1.088	1.168	0.985
		50	6	1547	1.095	1.097	1.071	1.131	1.066	1.144	0.985
		50	10	1965	1.080	1.098	1.062	1.126	1.056	1.133	0.988
		100	2	937	1.142	1.121	1.104	1.157	1.097	1.162	0.983
		100	6	1679	1.122	1.107	1.079	1.142	1.073	1.135	0.982
		100	10	2189	1.111	1.101	1.068	1.133	1.063	1.124	0.983
		200	2	974	1.158	1.133	1.112	1.168	1.105	1.158	0.982
		200	6	1797	1.140	1.118	1.086	1.150	1.080	1.129	0.980
		200	10	2389	1.131	1.111	1.075	1.141	1.069	1.117	0.980
100	0	10	2	797	1.034	1.115	1.093	1.120	1.088	1.311	1.025
		10	6	1205	1.015	1.152	1.079	1.114	1.076	1.276	1.041
		10	10	1411	1.012	1.197	1.069	1.108	1.067	1.256	1.052
		50	2	951	1.089	1.106	1.095	1.119	1.086	1.262	0.981
		50	6	1633	1.069	1.105	1.076	1.109	1.070	1.231	0.985
		50	10	2068	1.055	1.109	1.067	1.105	1.061	1.215	0.988
		100	2	995	1.114	1.118	1.106	1.130	1.096	1.250	0.977
		100	6	1774	1.096	1.108	1.083	1.117	1.076	1.215	0.977
		100	10	2306	1.087	1.107	1.072	1.111	1.065	1.196	0.978
		200	2	1033	1.131	1.130	1.115	1.140	1.106	1.245	0.975
		200	6	1896	1.116	1.117	1.092	1.126	1.083	1.206	0.974
		200	10	2515	1.106	1.111	1.080	1.117	1.071	1.182	0.974
200	0	10	2	857	1.030	1.125	1.114	1.110	1.110	1.473	1.048
		10	6	1280	1.009	1.167	1.116	1.124	1.113	1.438	1.084
		10	10	1495	1.005	1.207	1.108	1.119	1.105	1.409	1.102
		50	2	1037	1.064	1.098	1.095	1.088	1.088	1.388	0.981
		50	6	1768	1.041	1.097	1.087	1.091	1.079	1.348	0.988
		50	10	2222	1.032	1.104	1.085	1.094	1.079	1.335	0.996
		100	2	1088	1.084	1.106	1.102	1.103	1.093	1.368	0.972
		100	6	1927	1.069	1.096	1.088	1.092	1.080	1.321	0.973
		100	10	2492	1.062	1.095	1.081	1.089	1.074	1.300	0.976
		200	2	1130	1.100	1.115	1.111	1.112	1.100	1.355	0.967
		200	6	2059	1.087	1.103	1.095	1.099	1.086	1.305	0.966
		200	10	2719	1.079	1.097	1.086	1.093	1.076	1.276	0.968
500	0	10	2	1034	1.025	1.109	1.101	1.096	1.097	1.678	1.053
		10	6	1464	1.003	1.210	1.161	1.158	1.158	1.714	1.159
		10	10	1685	1.001	1.266	1.169	1.167	1.166	1.696	1.203
		50	2	1233	1.040	1.087	1.086	1.079	1.079	1.592	0.985
		50	6	2031	1.023	1.114	1.110	1.102	1.102	1.590	1.019
		50	10	2535	1.019	1.126	1.114	1.107	1.107	1.575	1.030
		100	2	1296	1.052	1.087	1.087	1.077	1.078	1.556	0.968
		100	6	2264	1.041	1.083	1.081	1.072	1.073	1.508	0.973
		100	10	2908	1.034	1.083	1.079	1.070	1.070	1.487	0.977
		200	2	1351	1.066	1.089	1.089	1.079	1.079	1.528	0.956
		200	6	2428	1.055	1.080	1.080	1.070	1.070	1.469	0.957
		200	10	3178	1.048	1.077	1.076	1.066	1.066	1.442	0.959

Table 4.5: Performance of (F,Q,s,S) Policy and (F,Q,S) over the other policies for 12-item Problem in Table 3.2 when the inventory holding cost is less than the shortages cost

Problem Parameters				(F, Q, s, S)	Cost of Other Policies (Given as a Ratio of the Cost of (F, Q, s, S))							
A	π	p	h		(F,Q,S)	(mF,S)	(F,s,S)	(Q,S)	(Q,s,S)	(s,c,S)	(s,c,S)-C	LB
20	0	1000	200	18322	1.087	1.020	1.097	1.086	1.016	1.017	1.011	0.989
		1000	600	33823	1.063	1.011	1.109	1.061	1.009	1.009	1.014	0.991
		1000	1000	43011	1.052	1.010	1.123	1.051	1.008	1.009	1.005	0.991
		5000	200	26039	1.082	1.016	1.074	1.077	1.014	1.012	1.007	0.990
		5000	600	57068	1.054	1.012	1.065	1.051	1.009	1.007	1.005	0.992
		5000	1000	80408	1.042	1.009	1.064	1.037	1.006	1.005	1.002	0.992
		10000	200	29059	1.081	1.016	1.071	1.076	1.014	1.011	1.008	0.990
		10000	600	66735	1.050	1.013	1.057	1.046	1.008	1.006	1.006	0.993
		10000	1000	96927	1.038	1.009	1.053	1.033	1.005	1.004	1.001	0.991
		20000	200	31920	1.078	1.016	1.067	1.073	1.013	1.010	1.007	0.990
		20000	600	75966	1.047	1.013	1.052	1.043	1.007	1.004	1.003	0.992
		20000	1000	112800	1.032	1.008	1.047	1.029	1.004	1.003	1.001	0.992
50	0	1000	200	18651	1.077	1.026	1.086	1.077	1.023	1.036	*	0.985
		1000	600	34277	1.056	1.017	1.101	1.054	1.013	1.024	*	0.986
		1000	1000	43524	1.046	1.016	1.115	1.045	1.012	1.021	*	0.988
		5000	200	26420	1.074	1.025	1.069	1.071	1.021	1.025	*	0.985
		5000	600	57655	1.049	1.014	1.060	1.048	1.011	1.015	*	0.988
		5000	1000	81072	1.039	1.012	1.059	1.034	1.009	1.011	*	0.989
		10000	200	29453	1.074	1.024	1.065	1.070	1.021	1.024	*	0.985
		10000	600	67351	1.046	1.015	1.055	1.041	1.012	1.013	*	0.989
		10000	1000	97703	1.035	1.011	1.048	1.030	1.008	1.009	*	0.988
		20000	200	32342	1.071	1.023	1.065	1.067	1.019	1.022	*	0.985
		20000	600	76614	1.044	1.015	1.048	1.039	1.010	1.012	*	0.989
		20000	1000	113590	1.031	1.010	1.044	1.027	1.006	1.008	*	0.989
100	0	1000	200	19101	1.067	1.030	1.095	1.068	1.025	1.060	1.022	0.980
		1000	600	34861	1.048	1.021	1.110	1.050	1.017	1.041	1.015	0.982
		1000	1000	44162	1.040	1.018	1.124	1.041	1.015	1.037	1.013	0.986
		5000	200	26939	1.066	1.031	1.074	1.064	1.025	1.045	1.018	0.981
		5000	600	58437	1.044	1.018	1.063	1.039	1.014	1.027	1.011	0.984
		5000	1000	81993	1.035	1.015	1.062	1.031	1.011	1.022	1.008	0.985
		10000	200	30010	1.066	1.030	1.070	1.064	1.024	1.040	1.018	0.979
		10000	600	68171	1.042	1.019	1.057	1.037	1.014	1.025	1.012	0.985
		10000	1000	98669	1.032	1.014	1.052	1.029	1.011	1.018	1.007	0.985
		20000	200	32910	1.064	1.029	1.067	1.060	1.024	1.039	1.015	0.980
		20000	600	77499	1.039	1.018	1.053	1.035	1.013	1.021	1.011	0.986
		20000	1000	114600	1.031	1.014	1.045	1.027	1.010	1.015	1.007	0.986
200	0	1000	200	19823	1.056	1.035	1.082	1.058	1.029	1.094	*	0.973
		1000	600	35835	1.038	1.023	1.100	1.041	1.019	1.070	*	0.978
		1000	1000	45227	1.033	1.022	1.117	1.035	1.017	1.060	*	0.981
		5000	200	27800	1.056	1.035	1.066	1.054	1.028	1.074	*	0.973
		5000	600	59662	1.039	1.022	1.060	1.036	1.017	1.047	*	0.978
		5000	1000	83392	1.032	1.020	1.058	1.028	1.014	1.034	*	0.982
		10000	200	30897	1.057	1.036	1.064	1.054	1.029	1.065	*	0.973
		10000	600	69505	1.038	1.021	1.052	1.035	1.017	1.040	*	0.978
		10000	1000	100160	1.029	1.019	1.050	1.025	1.014	1.028	*	0.981
		20000	200	33831	1.055	1.036	1.063	1.052	1.028	1.059	*	0.973
		20000	600	78889	1.036	1.021	1.048	1.031	1.017	1.036	*	0.980
		20000	1000	116280	1.029	1.017	1.044	1.025	1.013	1.028	*	0.982
500	0	1000	200	21507	1.037	1.038	1.072	1.043	1.031	1.168	1.033	0.961
		1000	600	38022	1.026	1.029	1.099	1.033	1.023	1.128	1.026	0.970
		1000	1000	47629	1.024	1.028	1.121	1.032	1.022	1.112	1.025	0.975
		5000	200	29759	1.044	1.041	1.059	1.044	1.032	1.133	1.028	0.958
		5000	600	62581	1.030	1.027	1.053	1.028	1.020	1.085	1.019	0.968
		5000	1000	86847	1.027	1.022	1.055	1.025	1.015	1.063	1.014	0.970
		10000	200	32930	1.045	1.042	1.057	1.043	1.033	1.122	1.028	0.959
		10000	600	72597	1.031	1.028	1.050	1.028	1.020	1.072	1.019	0.967
		10000	1000	103830	1.026	1.022	1.045	1.023	1.016	1.057	1.013	0.972
		20000	200	35927	1.046	1.043	1.057	1.045	1.033	1.112	1.027	0.959
		20000	600	82095	1.030	1.026	1.044	1.027	1.020	1.072	1.021	0.970
		20000	1000	120110	1.025	1.022	1.041	1.024	1.016	1.049	1.012	0.974

Table 4.6: Performance of (F,Q,s,S) Policy and (F,Q,S) over the other policies for 12-item Problem in Table 3.2 when the inventory holding cost is greater than the shortages cost

a	λ	(F, Q, s, S)	Cost of Other Policies (Given as a Ratio of the Cost of (F, Q, s, S))			
			(F, Q, S)	(F, s, S)	(Q, s, S)	LB
0	20	819	1.042	1.063	1.044	0.898
	40	1156	1.036	1.052	1.035	0.909
	60	1414	1.030	1.045	1.030	0.917
	80	1630	1.028	1.042	1.027	0.923
20	20	1013	1.074	1.097	1.080	0.936
	40	1437	1.069	1.085	1.069	0.944
	60	1763	1.062	1.076	1.062	0.949
	80	2039	1.057	1.070	1.057	0.952
40	20	1167	1.080	1.105	1.091	0.957
	40	1662	1.078	1.093	1.079	0.958
	60	2038	1.073	1.087	1.073	0.963
	80	2357	1.069	1.081	1.068	0.965
60	20	1302	1.079	1.106	1.092	0.964
	40	1851	1.083	1.097	1.084	0.969
	60	2275	1.077	1.089	1.077	0.971
	80	2634	1.072	1.084	1.072	0.972

Table 4.7: Performance of (F, Q, s, S) and (F, Q, S) for identical items with different demand rates and specific-item ordering cost. Other Parameters, $n = 8$, $A = 150$, $L = 0.2$, $h = 6$, $p = 0$, and $\pi = 30$.

L	n	(F, Q, s, S)	Cost of Other Policies (Given as a Ratio of the Cost of (F, Q, s, S))			
			(F, Q, S)	(F, s, S)	(Q, s, S)	LB
0.2	2	1022	1.017	1.056	1.016	0.966
	4	1188	1.040	1.064	1.039	0.950
	6	1318	1.055	1.077	1.059	0.945
	8	1437	1.069	1.085	1.069	0.944
	10	1542	1.079	1.093	1.080	0.944
	12	1617	1.088	1.115	1.104	0.963
0.4	2	1066	1.011	1.043	1.010	0.973
	4	1249	1.026	1.046	1.026	0.959
	6	1390	1.039	1.057	1.042	0.955
	8	1521	1.049	1.061	1.049	0.954
	10	1635	1.057	1.068	1.057	0.953
	12	1716	1.066	1.088	1.079	0.973
0.6	2	1102	1.007	1.035	1.007	0.976
	4	1297	1.020	1.038	1.020	0.965
	6	1447	1.030	1.047	1.034	0.961
	8	1587	1.038	1.049	1.038	0.960
	10	1706	1.046	1.055	1.046	0.959
	12	1791	1.053	1.074	1.067	0.980

Table 4.8: Performance of (F, Q, s, S) and (F, Q, S) for identical items with different leadtime and number of items. Other Parameters, $\mu = 320$,$A = 150$,$a = 20$, $h = 6$, $p = 0$, and $\pi = 30$

Items Demand				(F, Q, s, S)	Cost of Other Policies (Given as a Ratio of the Cost of (F, Q, s, S))			
λ_1	λ_2	λ_3	λ_4		(F, Q, S)	(F, s, S)	(Q, s, S)	LB
80	80	80	80	1188	1.040	1.064	1.039	0.950
70	70	70	110	1185	1.040	1.065	1.039	0.951
60	60	60	140	1180	1.039	1.066	1.039	0.954
50	50	50	170	1170	1.039	1.069	1.039	0.958
40	40	40	200	1156	1.040	1.074	1.039	0.966
70	70	90	90	1187	1.040	1.065	1.039	0.951
60	60	100	100	1184	1.040	1.065	1.039	0.952
50	50	110	110	1180	1.040	1.066	1.039	0.953
40	40	120	120	1173	1.041	1.069	1.040	0.957
70	83.33	83.33	83.33	1187	1.040	1.064	1.040	0.950
60	86.67	86.67	86.67	1186	1.040	1.065	1.040	0.951
50	90	90	90	1185	1.040	1.065	1.040	0.952
40	93.33	93.33	93.33	1183	1.039	1.065	1.039	0.952
70	60	100	90	1185	1.040	1.065	1.039	0.951
70	50	110	90	1184	1.040	1.065	1.039	0.952
70	40	120	90	1180	1.040	1.067	1.040	0.954
70	30	130	90	1175	1.041	1.068	1.040	0.956
70	20	140	90	1168	1.042	1.069	1.041	0.958

Table 4.9: Performance of (F, Q, s, S) and (F, Q, S) for non-identical items. Other Parameters, $n = 4$, $A = 150$, $a = 20$, $h = 6$, $p = 0$, and $\pi = 30$

4.6 Conclusion

In this chapter, we consider two periodic review policies for solving stochastic joint replenishment problems (SJRPs) referred as to (F, Q, S) policy and (F, Q, s, S) policy. The proposed policies base the ordering decisions on the aggregate demand since last replenishment and the time between two successive replenishments. We develop expressions for the operating characteristics of the inventory system and construct the expected total cost for a unit Poisson demand process.

An extensive numerical conducts to study the sensitivity of the policy to numerous problem parameters and to assess the performance of the proposed policy over the existing policies. The numerical results show that the (F, Q, s, S) policy is able to dominate all other policies in literature because (F, s, S) policy is a special case of the (F, Q, s, S) policy when $Q = 1$, and (Q, s, S) policy is a special case of the (F, Q, s, S) policy when $F \to 0$. The proposed policies were compared to the existing policies in literature and numerical results show that the (F, Q, s, S) policy performs much better and provides significant savings over the existing policies. Also, the (F, Q, S) policy performs slightly better over the policies compared. This finding, we believe, may perhaps have important implications for manufacturing/distributions systems.

Chapter 5

Joint Replenishment Problems with Different Demand Classes

This chapter is concerned with the joint replenishment problem (JRP) with two different demand sources; that are, deterministic demand and stochastic demand. The deterministic demand must be satisfied immediately while the stochastic demand that cannot be met from stock is backordered. A new model for JRP with priority demand classes (DSRJP) is discussed. Also, we propose a periodic review policy, called $(mF, R + S)$ policy.

The remainder of this chapter is organized as follows. In section 5.1 we describe the basic model for JRP with different demand classes and discuss its importance. Typical assumptions and notations for the model and the model formulation for the model are discussed in Sections 5.2 and 5.3 respectively. A heuristic algorithm is presented in Section 5.4. In Section 5.5, we discuss the performance of the proposed heuristic. Section 5.6 provides conclusions.

5.1 Introduction

We consider a joint replenishment problem (JRP) where the inventory system faces two different demand classes, which are deterministic demand and stochastic demand. The deterministic demand must be satisfied immediately, while the stochastic demand that cannot be satisfied from stock is backordered and fulfills when the replenishment order is arrived. Assume that items reserved for deterministic demand are never used for stochastic demand. The problem is to minimise the expected total ordering, inventory holding and shortages costs per unit time.

A periodic review period policy, called $(mF, R + S)$ is proposed to determine when and how much to order for each demand classes. In this policy, the inventory position for an item i is reviewed periodically every m_iF time units and then bring the inventory position up to level $R_i + S_i$, where m_i is a positive integer. For stochastic demand, Atkins and Iygun (1988) propose periodic review policy (mF, S) where the items are replenished periodically regardless of their inventory positions. Fung and Ma (2001) propose an algorithm to obtain an optimal solution for deterministic demand (see Appendix A). Basically, the solution for deterministic version of JRP is optimal whereas the solution for the stochastic JRP is not. Therefore, the optimal solution for the deterministic demand is used to solve the problem where demand is generated from two sources, one deterministic and one stochastic. Then, we compare the solution obtained from the deterministic version of JRP to the solution obtained from the new heuristic. For the deterministic solution, the integer multiple times m_i for each item i is computed are based only on deterministic demand. However, a heuristic algorithm is proposed where the integer multiple times m_i for each item i is computed are based on both deterministic demand and stochastic demand.

In most of the literature on joint replenishment problem, demand is treated either as deterministic or stochastic. However, in reality in many applications demand is generated from several sources. More specifically, demand is generated from a deterministic source and a stochastic source. Therefore, this chapter will be addressing a new model as a starting point of the research for such problem.

5.2 Typical Assumptions of the Model

Through this chapter, we use the notation given below and the following typical assumption are made.

1. For each item, there are two sources of demand. The first is assumed to be independent, deterministic and constant, and the second independent, stochastic and stationary, respectively;
2. For each item, the deterministic demand must be satisfied immediately while the stochastic demand that cannot be met from stock is backordered, where a customer agrees to wait until his order is fulfilled;
3. For each item, the other relevant parameters are deterministic and constant over a planning horizon that is considered infinite for planning purposes; namely, procurement lead time, specific-item ordering cost, inventory holding cost, and shortages costs;
4. For each item, stock reserved for deterministic demand source are never used for stochastic demand source;
5. The joint replenishment of a subset of products or all products achieves some sort of economies of scale and the joint ordering cost is independent of both the items ordered and their number;
6. Supply is readily available

Notations:

Parameters

n : number of items,

d_i : deterministic demand rate for product i expressed in units per year,

λ_i : mean of Poisson distribution of demand for item i expressed in units per year,

L_i : replenishment lead time of item i expressed in unit of time per year,

h_i : cost of inventory holding of item i expressed in \$/units per year,

p_i : backorder cost of per unit and time unit for item i,

π_i : one-off shortage cost of per unit of item i,

A : joint ordering cost associated with each replenishment order (\$/order),

a_i : specific-item ordering cost of item i, incurred if items i is ordered in a replenishment order expressed in \$/order of item i.

Decision Variables

F : basic review period length for replenishment order expressed in units per year,

m_i : the integer time multiple of the basic review period F for item i,

T_i : review period for item i expressed in unit of time per year, where $T_i = m_i F$,

S_i : order-up-to for the stochastic source for item i.

5.3 Model Formulation

In this section, we derive the inventory holding cost in the case where demand is deterministic. Then, we derive the expected inventory holding cost in the case where stochastic demands are generated by Poisson processes and units are demanded one a time. For $(m_i F, R_i + S_i)$ policy, the replenishment order cycle for each item i is an integer multiple m_i of the replenishment order cycle F. Thus, the replenishment order cycle for item i is

$$T_i = m_i F \tag{5.1}$$

and the order quantity during cycle time for the deterministic source R_i for item i is

$$R_i = \lceil T_i d_i \rceil = \lceil m_i F d_i \rceil \tag{5.2}$$

where $\lceil \cdot \rceil$ denotes integer ceiling function. The total inventory holding costs C_H^D in the case where demand is deterministic are

$$C_H^D = \sum_{i=1}^{n} h_i \frac{R_i}{2} = \sum_{i=1}^{n} h_i \frac{\lceil m_i F d_i \rceil}{2} \tag{5.3}$$

For the case where demand is stochastic, the expected total inventory holding costs IC_H^S are

$$\begin{aligned} C_H^S &= \sum_{i=1}^{n} \frac{\sum_{k=0}^{S-1} G_i(S_i - k)}{T_i} && (5.4) \\ &= \sum_{i=1}^{n} \frac{\sum_{k=0}^{S-1} G_i(S_i - k)}{m_i F} && (5.5) \end{aligned}$$

and

$$C_i^s(T_i) = \frac{a_i + \sum_{k=0}^{S-1} G_i(S - k)}{T_i} \tag{5.6}$$

where $C_i^*(m_i F)$ is the cost per unit time for item i corresponding to a review period of $T_i = m_i F$, and item i only bears the item-specific ordering cost where demand is stochastic. where $G_i(\cdot)$ is the exptected inventory holding and shortages costs function per unit time for item i (is the equation (3.13) computed in chapter 3).

The total ordering costs C_O are

$$C_O = \frac{A + \sum_{i=1}^{n} a_i / m_i}{F} \tag{5.7}$$

The expected total cost $TC(F, m_1, ..., m_n)$ for $(m_i F, R_i + S_i)$ policy with a review period of F is approximately:

$$TC(F, m_1, ..., m_n) \quad = \quad C_O + C_H^D + C_H^S \tag{5.8}$$

The actual cost of $(m_i F, R_i + S_i)$ policy is less than that evaluated by expression (5.7), since the major setup cost A is not incurred in a review period in which none of the items is ordered.

5.4 Solution Approach

This section is mainly concerned with the simple heuristic to solve the JRP model where the system faces different demand classes. First, we obtain the time multiple m_i for each item i by solving a deterministic JRP with demand $d_i+\lambda_i$. Second, we compute the optimal S_i policy by using the second part of the algorithm of Zheng and Federgruen (1991) for each item i with review period $T_i = m_i F$. Finally, we compute the expected total cost, say $TC(F)$ corresponding to the initial value of F, and increment F by 0.01 in either direction. The heuristic is conducted by varying F until we see no improvement in the total cost $TC(F)$ as defined in (5.8). The solution approach is summarised in the following three steps.

Step 0: (Initialization step)

Solve the deterministic version of the JRP with demand $D_i = d_i+\lambda_i$ for each item i $(i = 1, ..., n)$. Then, set the common cycle for deterministic version to an initial review period F, and the integer time multiples corresponding to the common cycle to m_i^*, for each item i $(i = 1, ..., n)$. Some optimal algorithms are presented in Appendix A.

Step 1: (An Iterative step)

For each item $(i = 1, ..., n)$, compute the optimal $S_i^*$and the cost of stochastic version of JRP, say $C_i^{s*}(T_i)$, corresponding to the current value of the review period $T_i = m_i^* F$, as follow:

(a) Set $S_i^* = y_i^*$, i.e., a value y_i of that minimizes $G_i(y_i)$. Then, compute the initial best solution so far $C_i^{s*}(T_i) = C_i^*(T_i, S_i^*)$.

(b) Set $S_i = S_i^* + 1$. If $G_i(S_i) > C_i^{s*}(T_i, S_i^*)$, then stop as S_i^* are the optimal solution with the cost $C_i^{s*}(T_i, S_i^*)$; else, if $C_i^s(T_i, S_i) < C_i^{s*}(T_i, S_i^*)$; Set $S_i^* = S_i$ and repeat this step.

and compute the cost $C_i^*(T_i)$ for item i which is composed of the cost of stochastic version of JRP and the total inventory holding costs for the deterministic version of JRP $\sum_i h_i \frac{\lceil T_i d_i \rceil}{2}$.Then, compute the total cost for DSJRP, say $TC(F)$, corresponding to the initial value of F.

Step 2: (Improvement step)

Increment F by 0.01, in either direction, then repeat step 1, and update $TC(F)$ and F. Stop if there is no further improvemment in the total cost $TC(F)$ for the current value of F.

5.5 Performance Evaluation

In this section, we obtain the optimal F and m_i for each item i using the classic JRPs with deterministic demand, and then obtain the expected total cost $TC(F)$ (5.4) using these values. We denote that P^1 is the solution obtained by the deterministic JRP, and P^2 is the solution obtained from the $(mF, R + S)$ policy. We compare the performance of the P^1 to P^2 through a numerical results. First of all, in Table 5.1, data are presented and the optimal policy solutions are shown. The deterministic demands d are uniformly generated $U(15, 60)$.

Further numerical results are conducted on the 12-item problem for a wide range of parameters. We use the same data in Table 5.1, with different values for the parameters A, p, h and $\pi = 0$. The results are presented in Table 5.2. For the problems whose results are presented in Table 5.2, the value $A = \{20, 50, 100, 200, 500\}$, $p = \{10, 50, 100, 200\}$ and $h = \{2, 6, 10\}$. The results show that the $(mF, Q + S)$ policy P^2 dominates the deterministic version of JRP P^1. The P^2 policy performs well over the P^1 for all the problems. The results also indicate that the P^2 policy achieves savings of up to 5% over the P^1 policy. In Table 5.3, we consider high values of inventory holding costs and shortages costs. The table reveals that the P^2 policy achieves savings of up to 1% over the P^1 policy when the inventory holding costs and shortages are very large.

To further verify the robustness of the proposed policies, we consider that the deterministic demand rate is a half of the stochastic demand rate, that is, $d = 0.5\lambda$ for each item. The results presented in Table 5.4 with moderate values of inventory holding costs and shortages costs. The numerical results reveal that the P^2 policy performs significantly better than P^1, and the P^2 policy achieves savings of up to 15% over the P^1 policy. Also, The results with high values of inventory holding costs and shortages costs are presented in Table 5.5, the P^2 policy achieves savings of up to 6% over the P^1 policy.

For further analysis, we consider that the deterministic demand rate is a double of the stochastic demand rate, that is, $d = 2\lambda$ for each item. The results presented in Table 5.6 with moderate values of inventory holding costs and shortages costs. The numerical results reveal that the P^2 policy performs better than P^1, and the P^2 policy achieves savings of up to 2% over the P^1 policy. Also, The results with high values of inventory holding costs and shortages costs are presented in Table 5.7, the P^2 policy achieves savings of up to 5% over the P^1 policy. We can see that for the high values of the inventory holding costs and shortages costs, the P^2 performs better than the case in which the inventory holding costs and shortages costs are moderate.

Problem Parameters					P1[1] $T^* = 0.976$			P2[2] $T^* = 0.674$		
i	λ	d	L	a	m	Q	S	m	Q	S
1	40	60	0.2	10	1	59	47	1	41	35
2	35	30	0.5	10	1	30	53	1	21	42
3	40	50	0.2	20	1	49	47	1	34	35
4	40	55	0.1	20	1	54	43	1	38	31
5	40	45	0.2	40	1	44	47	1	31	35
6	20	35	1.5	20	1	35	53	1	24	48
7	20	30	1	40	1	30	42	1	21	37
8	20	15	1	40	2	30	60	2	21	48
9	28	20	1	60	2	40	83	2	27	66
10	20	25	1	60	2	49	60	2	34	48
11	20	30	1	80	2	59	60	2	41	48
12	20	50	1	80	1	49	42	2	68	48
Total Cost					1602			1540		

[1] is the optimal solution for deterministic version to solve joint replenishment problem where demand is Deterministic and stochastic demand (DSJRP),
[2] is the stochastic policy $(mF, R+S)$ to solve DSJRP.

Table 5.1: Data and Typical Results for deterministic Policy and stochastic Policy for DSJRP where Deterministic Demand D is Uniformly generated, $U(15, 60)$. Other data are $n = 12$, $A = 150$, $h = 2$, $p = 30$, and $\pi = 0$

Problem Parameters					
A	p	h	P1[1]	P2[2]	P1/P2
20	10	2	1269	1231	1.031
	10	6	2076	2055	1.010
	10	10	2595	2584	1.004
	50	2	1420	1362	1.042
	50	6	2461	2394	1.028
	50	10	3159	3095	1.021
	100	2	1476	1411	1.046
	100	6	2615	2532	1.033
	100	10	3403	3319	1.025
	200	2	1527	1457	1.048
	200	6	2757	2660	1.037
	200	10	3632	3533	1.028
50	10	2	1316	1282	1.026
	10	6	2159	2128	1.015
	10	10	2704	2694	1.004
	50	2	1468	1417	1.036
	50	6	2546	2481	1.026
	50	10	3269	3205	1.020
	100	2	1524	1468	1.038
	100	6	2701	2623	1.030
	100	10	3514	3430	1.025
	200	2	1576	1515	1.040
	200	6	2844	2754	1.033
	200	10	3744	3643	1.028
100	10	2	1400	1363	1.027
	10	6	2292	2261	1.014
	10	10	2865	2859	1.002
	50	2	1558	1503	1.037
	50	6	2693	2626	1.025
	50	10	3452	3389	1.018
	100	2	1616	1555	1.039
	100	6	2852	2769	1.030
	100	10	3702	3616	1.024
	200	2	1669	1603	1.041
	200	6	2999	2904	1.033
	200	10	3939	3833	1.028
200	10	2	1552	1505	1.031
	10	6	2526	2493	1.013
	10	10	3151	3125	1.008
	50	2	1721	1650	1.043
	50	6	2957	2873	1.029
	50	10	3780	3707	1.020
	100	2	1783	1704	1.046
	100	6	3125	3024	1.033
	100	10	4047	3944	1.026
	200	2	1839	1754	1.049
	200	6	3280	3164	1.036
	200	10	4295	4167	1.031
500	10	2	1834	1785	1.027
	10	6	2985	2951	1.012
	10	10	3723	3722	1.001
	50	2	2017	1944	1.038
	50	6	3452	3359	1.028
	50	10	4405	4312	1.022
	100	2	2083	2002	1.040
	100	6	3632	3520	1.032
	100	10	4689	4562	1.028
	200	2	2142	2055	1.042
	200	6	3795	3670	1.034
	200	10	4950	4800	1.031

[1] is the optimal solution for deterministic version to solve joint replenishment problem where demand is Deterministic and stochastic demand (DSJRP),
[2] is the stochastic policy $(mF, R + S)$ to solve DSJRP.

Table 5.2: Performance of $(mF, R+S)$ Policy over Deterministic Policy for 12-item Problem in Table 5.1

Problem Parameters			P1[1]	P2	P1/P2
A	p	h			
20	1000	200	25376	25124	1.010
	1000	600	48214	47420	1.017
	1000	1000	61426	61426	1.000
	5000	200	33243	32864	1.012
	5000	600	71579	70505	1.015
	5000	1000	98632	98632	1.000
	10000	200	36352	36001	1.010
	10000	600	81352	80125	1.015
	10000	1000	115336	115336	1.000
	20000	200	39342	38903	1.011
	20000	600	90585	89397	1.013
	20000	1000	131330	131330	1.000
50	1000	200	25893	25787	1.004
	1000	600	48311	47939	1.008
	1000	1000	64071	63568	1.008
	5000	200	33782	33484	1.009
	5000	600	71616	71080	1.008
	5000	1000	101185	100486	1.007
	10000	200	36897	36535	1.010
	10000	600	81401	80668	1.009
	10000	1000	117672	117136	1.005
	20000	200	39840	39425	1.011
	20000	600	90627	89925	1.008
	20000	1000	133838	132890	1.007
100	1000	200	26630	26602	1.001
	1000	600	48974	48974	1.000
	1000	1000	64799	64600	1.003
	5000	200	34593	34389	1.006
	5000	600	72300	72300	1.000
	5000	1000	102073	101554	1.005
	10000	200	37707	37483	1.006
	10000	600	82224	82224	1.000
	10000	1000	118628	118175	1.004
	20000	200	40740	40386	1.009
	20000	600	91493	91493	1.000
	20000	1000	134721	134038	1.005
200	1000	200	28258	27945	1.011
	1000	600	51009	51009	1.000
	1000	1000	67097	66982	1.002
	5000	200	36347	35794	1.015
	5000	600	74552	74552	1.000
	5000	1000	104682	104392	1.003
	10000	200	39540	38876	1.017
	10000	600	84524	84524	1.000
	10000	1000	121346	121186	1.001
	20000	200	42564	41882	1.016
	20000	600	94130	93980	1.002
	20000	1000	137483	137397	1.001
500	1000	200	30617	30167	1.015
	1000	600	55809	55664	1.003
	1000	1000	71085	71085	1.000
	5000	200	38878	38339	1.014
	5000	600	79714	79227	1.006
	5000	1000	108954	108954	1.000
	10000	200	42197	41741	1.011
	10000	600	89777	89125	1.007
	10000	1000	125932	125932	1.000
	20000	200	45297	44750	1.012
	20000	600	99333	98551	1.008
	20000	1000	142311	142311	1.000

[1] is the optimal solution for deterministic version to solve joint replenishment problem where demand is Deterministic and stochastic demand (DSJRP),
[2] is the stochastic policy $(mT, S+Q)$ to solve DSJRP.

Table 5.3: Performance of $(mF, R+S)$ Policy over Deterministic Policy for 12-item Problem in Table 5.1 with High Values of Inventory Holding and Shortages Costs

Problem Parameters			P1[1]	P2	P1/P2
A	p	h			
20	10	2	1087	985	1.104
	10	6	1679	1576	1.065
	10	10	2033	1972	1.031
	50	2	1277	1130	1.130
	50	6	2163	1967	1.100
	50	10	2737	2518	1.087
	100	2	1345	1184	1.136
	100	6	2347	2117	1.109
	100	10	3027	2753	1.099
	200	2	1406	1234	1.140
	200	6	2516	2258	1.114
	200	10	3297	2978	1.107
50	10	2	1139	1026	1.110
	10	6	1766	1647	1.072
	10	10	2096	2039	1.028
	50	2	1334	1178	1.133
	50	6	2264	2039	1.110
	50	10	2821	2610	1.081
	100	2	1404	1236	1.135
	100	6	2454	2195	1.118
	100	10	3119	2855	1.092
	200	2	1467	1287	1.140
	200	6	2626	2339	1.123
	200	10	3395	3087	1.100
100	10	2	1214	1091	1.113
	10	6	1876	1758	1.067
	10	10	2233	2180	1.025
	50	2	1416	1247	1.136
	50	6	2393	2150	1.113
	50	10	2988	2753	1.086
	100	2	1488	1304	1.141
	100	6	2588	2305	1.123
	100	10	3295	2998	1.099
	200	2	1553	1357	1.145
	200	6	2766	2450	1.129
	200	10	3579	3231	1.108
200	10	2	1325	1195	1.109
	10	6	2042	1920	1.063
	10	10	2471	2391	1.033
	50	2	1539	1357	1.135
	50	6	2592	2337	1.109
	50	10	3274	2992	1.094
	100	2	1614	1416	1.140
	100	6	2796	2501	1.118
	100	10	3595	3247	1.107
	200	2	1682	1470	1.144
	200	6	2982	2652	1.124
	200	10	3892	3488	1.116
500	10	2	1582	1426	1.110
	10	6	2429	2296	1.058
	10	10	2917	2830	1.031
	50	2	1821	1603	1.136
	50	6	3048	2745	1.111
	50	10	3825	3502	1.092
	100	2	1903	1667	1.142
	100	6	3273	2919	1.121
	100	10	4180	3781	1.105
	200	2	1977	1725	1.146
	200	6	3475	3078	1.129
	200	10	4501	4042	1.114

[1] is the optimal solution for deterministic version to solve joint replenishment problem where demand is Deterministic and stochastic demand (DSJRP),
[2] is the stochastic policy $(mF, R + S)$ to solve DSJRP.

Table 5.4: Performance of $(mF, R+S)$ Policy over Deterministic Policy for 12-item Problem in Table 5.1 where Deterministic Demand $D = \alpha\lambda$, where $\alpha = 0.5$

Problem Parameters			P1[1]	P2	P1/P2
A	p	h			
20	1000	200	22566	22540	1.001
	1000	600	43058	41966	1.026
	1000	1000	54374	54135	1.004
	5000	200	30831	30424	1.013
	5000	600	66976	65398	1.024
	5000	1000	92281	92181	1.001
	10000	200	34118	33601	1.015
	10000	600	77056	75220	1.024
	10000	1000	109176	109111	1.001
	20000	200	37241	36556	1.019
	20000	600	86546	84593	1.023
	20000	1000	125511	125127	1.003
50	1000	200	23402	22531	1.039
	1000	600	43110	42804	1.007
	1000	1000	57209	54293	1.054
	5000	200	31795	30542	1.041
	5000	600	67201	66101	1.017
	5000	1000	95366	91777	1.039
	10000	200	35098	33710	1.041
	10000	600	77213	75781	1.019
	10000	1000	112363	108366	1.037
	20000	200	38248	36755	1.041
	20000	600	86873	85016	1.022
	20000	1000	128644	124549	1.033
100	1000	200	23826	22977	1.037
	1000	600	43932	43482	1.010
	1000	1000	58316	55076	1.059
	5000	200	32289	31015	1.041
	5000	600	68206	66786	1.021
	5000	1000	96595	92684	1.042
	10000	200	35683	34244	1.042
	10000	600	78206	76567	1.021
	10000	1000	113884	109294	1.042
	20000	200	38791	37281	1.040
	20000	600	87873	85794	1.024
	20000	1000	130040	125484	1.036
200	1000	200	25088	24167	1.038
	1000	600	44036	44036	1.000
	1000	1000	58507	58364	1.002
	5000	200	33755	32136	1.050
	5000	600	68468	68461	1.000
	5000	1000	97059	95741	1.014
	10000	200	37154	35311	1.052
	10000	600	78962	78800	1.002
	10000	1000	114174	112339	1.016
	20000	200	40451	38314	1.056
	20000	600	88687	88442	1.003
	20000	1000	130885	128405	1.019
500	1000	200	27317	26431	1.034
	1000	600	48570	46788	1.038
	1000	1000	64295	60328	1.066
	5000	200	36420	34686	1.050
	5000	600	73794	71022	1.039
	5000	1000	103979	99207	1.048
	10000	200	39904	38002	1.050
	10000	600	84289	81384	1.036
	10000	1000	121412	116591	1.041
	20000	200	43293	41104	1.053
	20000	600	94420	90895	1.039
	20000	1000	138179	133219	1.037

[1] is the optimal solution for deterministic version to solve joint replenishment problem where demand is Deterministic and stochastic demand (DSJRP),
[2] is the stochastic policy $(mF, R+S)$ to solve DSJRP.

Table 5.5: Performance of $(mF, R+S)$ Policy over Deterministic Policy for 12-item Problem in Table 5.1 with High Values of Inventory and Shortages Costs where Deterministic Demand $D = \alpha\lambda$, where $\alpha = 0.5$

Problem Parameters			P1[1]	P2	P1/P2
A	p	h			
20	10	2	1420	1404	1.011
	10	6	2376	2367	1.004
	10	10	3003	3003	1.000
	50	2	1555	1527	1.019
	50	6	2719	2690	1.011
	50	10	3511	3494	1.005
	100	2	1606	1573	1.021
	100	6	2859	2822	1.013
	100	10	3732	3703	1.008
	200	2	1653	1616	1.023
	200	6	2990	2939	1.017
	200	10	3944	3904	1.010
50	10	2	1480	1462	1.013
	10	6	2488	2460	1.012
	10	10	3124	3124	1.000
	50	2	1619	1589	1.019
	50	6	2840	2788	1.019
	50	10	3641	3614	1.008
	100	2	1670	1636	1.021
	100	6	2983	2921	1.021
	100	10	3868	3827	1.011
	200	2	1718	1681	1.022
	200	6	3116	3046	1.023
	200	10	4082	4029	1.013
100	10	2	1580	1559	1.014
	10	6	2646	2625	1.008
	10	10	3339	3314	1.008
	50	2	1723	1686	1.022
	50	6	3008	2957	1.017
	50	10	3870	3814	1.015
	100	2	1776	1735	1.024
	100	6	3154	3091	1.020
	100	10	4101	4032	1.017
	200	2	1825	1779	1.025
	200	6	3290	3218	1.023
	200	10	4322	4241	1.019
200	10	2	1733	1708	1.015
	10	6	2880	2873	1.002
	10	10	3674	3652	1.006
	50	2	1882	1845	1.020
	50	6	3259	3221	1.012
	50	10	4232	4174	1.014
	100	2	1937	1896	1.022
	100	6	3412	3362	1.015
	100	10	4472	4399	1.017
	200	2	1988	1943	1.023
	200	6	3552	3494	1.017
	200	10	4699	4614	1.018
500	10	2	2070	2043	1.013
	10	6	3449	3423	1.008
	10	10	4349	4342	1.002
	50	2	2234	2193	1.019
	50	6	3865	3804	1.016
	50	10	4959	4903	1.011
	100	2	2293	2248	1.020
	100	6	4029	3956	1.018
	100	10	5218	5144	1.014
	200	2	2348	2299	1.021
	200	6	4180	4097	1.020
	200	10	5460	5373	1.016

[1] is the optimal solution for deterministic version to solve joint replenishment problem where demand is Deterministic and stochastic demand (DSJRP),
[2] is the stochastic policy $(mF, R+S)$ to solve DSJRP.

Table 5.6: Performance of $(mF, R+S)$ Policy over Deterministic Policy for 12-item Problem in Table 5.1 where Deterministic Demand $D = \alpha\lambda$, where $\alpha = 2$

Problem Parameters			P1[1]	P2	P1/P2
A	p	h			
20	1000	200	27105	26967	1.005
	1000	600	51510	50469	1.021
	1000	1000	66029	66029	1.000
	5000	200	34819	34704	1.003
	5000	600	74637	73412	1.017
	5000	1000	102926	102926	1.000
	10000	200	37876	37756	1.003
	10000	600	84309	83009	1.016
	10000	1000	119493	119493	1.000
	20000	200	40767	40663	1.003
	20000	600	93608	92259	1.015
	20000	1000	135416	135416	1.000
50	1000	200	27434	27344	1.003
	1000	600	50890	50890	1.000
	1000	1000	69208	66365	1.043
	5000	200	35146	35020	1.004
	5000	600	74134	74134	1.000
	5000	1000	106164	103221	1.029
	10000	200	38243	37977	1.007
	10000	600	83675	83675	1.000
	10000	1000	122782	119839	1.025
	20000	200	41146	40883	1.006
	20000	600	92909	92909	1.000
	20000	1000	138653	135519	1.023
100	1000	200	28900	28889	1.000
	1000	600	54639	53810	1.015
	1000	1000	70685	67522	1.047
	5000	200	36648	36572	1.002
	5000	600	77939	76871	1.014
	5000	1000	107687	104413	1.031
	10000	200	39797	39569	1.006
	10000	600	87519	86505	1.012
	10000	1000	124255	121060	1.026
	20000	200	42694	42469	1.005
	20000	600	96733	95782	1.010
	20000	1000	140270	136750	1.026
200	1000	200	29916	29916	1.000
	1000	600	54407	54174	1.004
	1000	1000	74258	73170	1.015
	5000	200	37799	37714	1.002
	5000	600	77645	77433	1.003
	5000	1000	111434	110447	1.009
	10000	200	40892	40850	1.001
	10000	600	87503	87303	1.002
	10000	1000	127951	126968	1.008
	20000	200	43897	43732	1.004
	20000	600	96775	96601	1.002
	20000	1000	143947	142910	1.007
500	1000	200	33124	32950	1.005
	1000	600	60168	60168	1.000
	1000	1000	80182	79025	1.015
	5000	200	41165	40850	1.008
	5000	600	83898	83898	1.000
	5000	1000	117646	116239	1.012
	10000	200	44384	43941	1.010
	10000	600	93716	93716	1.000
	10000	1000	134494	132871	1.012
	20000	200	47407	46954	1.010
	20000	600	103189	103189	1.000
	20000	1000	150781	148808	1.013

[1] is the optimal solution for deterministic version to solve joint replenishment problem where demand is Deterministic and stochastic demand (DSJRP),
[2] is the stochastic policy $(mT, R+S)$ to solve DSJRP.

Table 5.7: Performance of $(mF, S+Q)$ Policy over Deterministic Policy for 12-item Problem in Table 5.1 with High Values of Inventory and Shortages Costs where Deterministic Demand $D = \alpha\lambda$, where $\alpha = 2$

5.6 Conclusion

In this chapter, we consider a new JRP model for addressing joint replenishment problems (JRPs) where inventory system faces two different demand classes. This JRP model deals with the problem when demand arises from deterministic source and stochastic source. A new period review $(mF, R+S)$ policy is proposed. The proposed policy assumes a basic review period for all items and the review period of each items is an integer multiple of a basic review period.

We compared the solutions obtained by deterministic version of JRPs to address the problem to the new heuristic algorithm. The results obtained reveal that the heuristic algorithm is able to handle this problem efficiently and provides significant savings over the deterministic solution. This finding, we believe, may perhaps have important implications for manufacturing/distributions systems.

Chapter 6

Summary, Conclusions, and Future Research

In this chapter, the contributions of this research is summarised and some future research directions is provided.

6.1 Summary and Conclusions

This thesis deals with joint replenishment problems (JRPs) which are part of multi-item lot sizing and scheduling problems in manufacturing and distribution systems in single echelon/stage systems. The particular problem we consider the stochastic joint replenishment problem in a single-location/multi-product and single-product/multi-location inventory system settings. The objective of the problem is to determine the optimal replenishment policy to minimise the expected total costs which is composed of ordering costs, inventory holding costs, and shortages costs. Demands of the items are stochastic and stationary. Because of the applicability to inventory control systems, stochastic joint replenishment problem (SJRPs) is a challenging research area. There are a few policies proposed in literature for the stochastic joint replenishment problems (SJRPs). A common measure used in the comparison is the ratio of other policies to the proposed policy. We propose three new periodic review policies for the stochastic joint replenishment problems. Also, because of the lack of research on joint replenishment problems with different demand classes (DSJPRs), we propose a new model for DSJRPs.

we propose periodic review policy for SJRPs, which is referred to as (mF, s, S) policy. In addition, a heuristic Algorithm has been developed to search for near optimal parameters for an (mF, s, S), (mF, S)-2 and (mF, S)-1 policy proposed by Atkins and Iyogun (1988). Numerical tests against literature benchmarks have been presented. Numerical results show that (mF, s, S) dominates all the policies compared specially when the inventory holding costs are greater than the shortages costs. Also, the performance of (mF, s, S) and $(mF, S$-2 policy remain on the same level as (F, s, S) policy and even becomes slightly better when the inventory holding costs are less than the shortages costs. We investigate the effects of the non-identical parameters; that are, the inventory holding cost, and the shortages costs on the periodic review policies. However, similarity of items in their replenishment lead time appears to be most critical factor in the dominance of the proposed policy. We find that the proposed policy provides significant savings over the existing policies for items similar in their replenishment lead time.

On the other hand, we propose two new periodic review policies for solving stochastic joint replenishment problems (SJRPs) referred as to (F, Q, S) policy and (F, Q, s, S) policy. The proposed policies base the replenishment decisions on the total demand that have accumulated for all items during a replenishment cycle. For these policies, we developed expressions for the operating characteristics of the inventory system and constructed the expected total cost function for Poisson demand process. The numerical results show that the (F, Q, s, S) policy is able to dominate all other policies in literature because (F, s, S) policy is a special case of the (F, Q, s, S) policy when $Q = 1$, and (Q, s, S) policy is a special case of the (F, Q, s, S) policy when F , , 0. The proposed policies were compared to the existing policies in literature and numerical results show that the (F, Q, s, S) policy performs much better and provides significant savings over the existing policies. Also, the (F, Q, S) policy performs slightly better over the policies compared. This finding, we believe, may have important implications for manufacturing/distributions systems.

Finally, we consider a new JRP model for addressing joint replenishment problems (JRPs) where inventory system faces two different demand classes (DSJRPs). This JRP model deals with the problem when demand comes from deterministic source and stochastic source. A period review $(mF, R+S)$ policy is proposed. That policy would specify how much of the demand from the stochastic source should be satisfied immediately and how much should be backordered. As the literature on inventory and production systems treat demand either as a deterministic pattern or as a stochastic pattern, this study herein addresses the DSJRPs. The comparison was made, on the basis of expected total cost, by how well the policy proposed perform relative to optimal solution for deterministic JRPs. The results obtained reveal that the policy proposed is able to handle this problem efficiently and provides significant savings over the optimal solution for deterministic JRPs. The model and findings have essential implications for manufacturing and distribution systems as well.

6.2 Future research

The basic aim of this thesis is to develop a basic analytical model for a class of joint replenishment policies. In this section, we provide possible research extensions. We basically consider a joint replenishment problem under single-location/multi-items or single-item/multi-location. The strategies used for stochastic joint replenishment problems (SJRPs) are indirect grouping strategies (IGS), where opportunity replenishment period is made at fixed time-periods and each item has a replenishment order sufficient to last for an integer multiple of the fixed time period. In those strategies, it does not synchronize transportation with replenishment. An important extension of the study presented in this thesis is to use direct grouping strategies (DGS) for stochastic joint replenishment problems (SJRPs), which a group is defined as the set of items that have the same opportunity replenishment period. Consequently, items of the same group are jointly replenished. The main challenge of direct grouping strategies is to determine the number of items into a certain number of different groups. We also need to assess the performance of the direct grouping strategies (DGS) and compare the results with the indirect grouping strategies (IGS).

Obviously, this thesis has highlighted numerous avenues for future research of the SJRPs. Despite the fact research on SJRPs may have received less attentions, there are several practical extensions which are interesting issues to focus on. The extensions typically deal with the SJRPs with resource restriction such as storage, transport capacities, budget, and other resource constraints (Hoque, 2006; Moon and Cha, 2006; Porras and Dekker, 2006). Other extensions of the SJRPs which are useful to explore is that one can consider multi-echelon production/inventory systems which extremely hard to address. Also, another potential future research direction would be to analyse the problem and design solution procedures for SJRPs under different explicit assumptions such as the independence of demands and stochastic replenishment lead time.

On the other hand, in this study we start with a new model for JRPs where the inventory system faces two different demand classes (DSJRPs) where inventory faces demands from deterministic source and stochastic source. We suggest that an $(mF, R + S)$ policy to address the DSJRPs. Thus, one can adopt a replenishment policy for the SJRPs to address the problem and determine how these perform with respect to the existing policies. This study has just begun to explore the applications of DSJRPs to the inventory control research.

Appendix A

Description of Optimal Solution Procedures

1. Fung and Ma (2001) Procedure (FMO1)

Step 1: Number items in the ascending order of their ratio $a_i/h_i d_i$ and set the time multiple of item number 1 to 1; that is, $m_1 = 1$ - note that the basic idea behind this step lies in the fact that if $a_i/h_i d_i \leq a_j/h_j d_j$, then $\tilde{m}_i \leq \tilde{m}_j$, where $\tilde{m}_i$ and $\tilde{m}_j$ denote the continuous approximations of m_i and m_j, respectively. For each item i ($i = 2, \ldots, n$), compute the continuous approximation of time multiple m_i as follows:

$$\tilde{m}_i = \sqrt{\frac{a_i}{h_i d_i} \cdot \frac{h_1 d_1}{A + a_1}},$$

and round it to the nearest integer greater than zero, say m_i. Then compute the corresponding total cost, say TC, as follows:

$$TC = \sqrt{2\left(A + \sum_{i=1}^{n} \frac{a_i}{m_i}\right) \cdot \left(\sum_{i=1}^{n} h_i d_i m_i\right)},$$

and initialize the upper bound on the basic period F, say F_{UB}, to $TC/\sum_{i=1}^{n} h_i d_i m_i$. Initialize the best solution found so far, say $(F^*, m_1^*, ..., m_n^*)$, and the corresponding total cost, say TC^*, as follows: $F^* = F_{UB}$, $m_i^* = m_i$ for all i, and $TC^* = TC$.

Determine the vector of time multiples corresponding to the current value of F_{UB} so that:
$m_i^{UB}\left(m_i^{UB}-1\right) \le \left(T_i^*/F_{UB}\right)^2 \le m_i^{UB}\left(m_i^{UB}+1\right)$ for all i,
and compute the corresponding total cost, say TC_{UB}. If $TC_{UB} \le TC^*$, then update the best solution found so far and update F_{UB} to $TC^* \Big/ \sum_{i=1}^n h_i d_i m_i^{UB}$.

Step 2: Initialize the lower bound on the basic period F, say F_{LB}, to $2s/TC^*$– notice that if the optimal strict cyclic policy is required, F_{LB} would be set to $\max\left\{\min_i \sqrt{2a_i/h_i d_i}, 2\left(A+\min_i a_i\right)\Big/TC^*\right\}$. Determine the vector of time multiples corresponding to the current value of F_{LB} so that:
$m_i^{LB}\left(m_i^{LB}-1\right) \le \left(T_i^*/F_{LB}\right)^2 \le m_i^{LB}\left(m_i^{LB}+1\right)$ for all i,
and compute the corresponding total cost, say TC_{LB}. If $TC_{LB} \le TC^*$, then update the best solution found so far and update F_{LB} to $2\left(A+\sum_{i=1}^n \frac{a_i}{m_i^{LB}}\right)\Big/TC^*$– notice that if the optimal strict cyclic policy is required, F_{LB} would be set to $2\left(A+a_j+\sum_{\substack{i=1\\ i\neq j}}^n \frac{a_i}{m_i^{LB}}\right)\Big/TC^*$, where j denotes the index of the item with the smallest optimal independent cycle time.

Step 3: For each item i ($i = 1,\ldots,n$), compute the value of the basic period at which the value of the time multiple m_i^{UB} increases by 1; that is:
$F_i = T_i^* \Big/ \sqrt{m_i^{UB}\left(m_i^{UB}+1\right)}$, where $T_i^* = \sqrt{2a_i/h_i d_i}$.

Step 4: Set the basic period $F = \max_i F_i$. If $F \le F_{LB}$, then stop; else, goto Step 5.

Step 5: Let p denotes the item for which m_i^{UB} changes to $m_i^{UB}+1$ at time F. Increase m_p^{UB} by 1 and update F_p accordingly. Compute the total cost TC corresponding to the modified vector of time multiples. If $TC \le TC^*$, then update the best solution found so far accordingly, update F_{LB} to $2\left(A+\sum_{i=1}^n \frac{a_i}{m_i^{LB}}\right)\Big/TC^*$– notice that if the optimal strict cyclic policy is required, F_{LB} would be set to $2\left(A+a_j+\sum_{\substack{i=1\\ i\neq j}}^n \frac{a_i}{m_i^{LB}}\right)\Big/TC^*$, where j denotes the index of the item with the smallest optimal independent cycle time, and goto Step 4; else, stop.

2. Fung and Ma (2001) Procedure (FMO2)

Step 1: Number items in the ascending order of their ratio $a_i/h_i d_i$ and set the time multiple of item number 1 to 1; that is, $m_1 = 1$ - note that the basic idea behind this step lies in the fact that if $a_i/h_i d_i \leq a_j/h_j d_j$, then $\tilde{m}_i \leq \tilde{m}_j$, where $\tilde{m}_i$ and $\tilde{m}_j$ denote the continuous approximations of m_i and m_j, respectively. For each item i ($i = 2, \ldots, n$), compute the continuous approximation of time multiple m_i as follows:

$$\tilde{m}_i = \sqrt{\frac{a_i}{h_i d_i} \cdot \frac{h_1 d_1}{A + a_1}},$$

and round it to the nearest integer greater than zero, say m_i. Then compute the corresponding total cost, say TC, as follows:

$$TC = \sqrt{2\left(A + \sum_{i=1}^{n} \frac{a_i}{m_i}\right) \cdot \left(\sum_{i=1}^{n} h_i d_i m_i\right)},$$

and initialize the upper bound on the basic period F, say F_{UB}, to $TC/\sum_{i=1}^{n} h_i d_i m_i$. Initialize the best solution found so far, say $(F^*, m_1^*, ..., m_n^*)$, and the corresponding total cost, say TC^*, as follows: $F^* = F_{UB}$, $m_i^* = m_i$ for all i, and $TC^* = TC$. Determine the vector of time multiples corresponding to the current value of F_{UB} so that:

$m_i^{UB}\left(m_i^{UB} - 1\right) \leq \left(T_i^*/F_{UB}\right)^2 \leq m_i^{UB}\left(m_i^{UB} + 1\right)$ for all i,

and compute the corresponding total cost, say TC_{UB}. If $TC_{UB} \leq TC^*$, then update the best solution found so far, update F_{UB} to $TC^*/\sum_{i=1}^{n} h_i d_i m_i^{UB}$ and repeat this step; else, goto the next step.

Step 2: Initialize the lower bound on the basic period F, say F_{LB}, to $2s/TC^*$– notice that if the optimal strict cyclic policy is required, F_{LB} would be set to $\max\left\{\min_i \sqrt{2a_i/h_i d_i}, 2\left(A + \min_i a_i\right)/TC^*\right\}$. Determine the vector of time multiples corresponding to the current value of F_{LB} so that:

$m_i^{LB}\left(m_i^{LB} - 1\right) \leq \left(T_i^*/F_{LB}\right)^2 \leq m_i^{LB}\left(m_i^{LB} + 1\right)$ for all i,

and compute the corresponding total cost, say TC_{LB}. If $TC_{LB} \leq TC^*$, then update the best solution found so far, update F_{LB} to $2\left(A + \sum_{i=1}^{n} \frac{a_i}{m_i^{LB}}\right)/TC^*$ – notice that if the optimal strict cyclic policy is required, F_{LB} would be set to $2\left(A + a_j + \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{a_i}{m_i^{LB}}\right)/TC^*$, where j denotes the index of the item with the smallest optimal independent cycle time, and repeat this step; else, goto the next step.

Step 3: For each item i ($i = 1,\ldots,n$), compute the value of the basic period at which the value of the time multiple m_i^{UB} increases by 1; that is:
$F_i = T_i^* \Big/ \sqrt{m_i^{UB}\left(m_i^{UB}+1\right)}$, where $T_i^* = \sqrt{2a_i/h_i d_i}$.

Step 4: Set the basic period $F = \max_i F_i$. If $F \leq F_{LB}$, then stop; else, goto Step 5.

Step 5: Let p denotes the item for which m_i^{UB} changes to $m_i^{UB}+1$ at time F. Increase m_p^{UB} by 1 and update F_p accordingly. Compute the total cost TC corresponding to the modified vector of time multiples. If $TC \leq TC^*$, then update the best solution found so far accordingly, update F_{LB} to $2\left(A+\sum_{i=1}^{n}\frac{a_i}{m_i^{LB}}\right)\Big/TC^*$– notice that if the optimal strict cyclic policy is required, F_{LB} would be set to $2\left(A+a_j+\sum_{\substack{i=1\\i\neq j}}^{n}\frac{a_i}{m_i^{LB}}\right)\Big/TC^*$, where j denotes the index of the item with the smallest optimal independent cycle time, and goto Step 4; else, stop.

3. Viswanathan (2002) Procedure (VO2) – Modified FMO2 for Strict Cyclic Policy

Step 1: Number items in the ascending order of their ratio $a_i/h_i d_i$ and set the time multiple of item number 1 to 1; that is, $m_1 = 1$ - note that the basic idea behind this step lies in the fact that if $a_i/h_i d_i \leq a_j/h_j d_j$, then $\tilde{m}_i \leq \tilde{m}_j$, where $\tilde{m}_i$ and $\tilde{m}_j$ denote the continuous approximations of m_i and m_j, respectively. For each item i ($i = 2, \ldots, n$), compute the continuous approximation of time multiple m_i as follows:

$$\tilde{m}_i = \sqrt{\frac{a_i}{h_i d_i}\cdot\frac{h_1 d_1}{A+a_1}},$$

and round it to the nearest integer greater than zero, say m_i. Then compute the corresponding total cost, say TC, as follows:

$$TC = \sqrt{2\left(A+\sum_{i=1}^{n}\frac{a_i}{m_i}\right)\cdot\left(\sum_{i=1}^{n}h_i d_i m_i\right)},$$

and initialize the upper bound on the basic period F, say F_{UB}, to $TC/\sum_{i=1}^{n}h_i d_i m_i$. Initialize the best solution found so far, say $(F^*, m_1^*, ..., m_n^*)$, and the corresponding total cost, say TC^*, as follows: $F^* = F_{UB}$, $m_i^* = m_i$ for all i, and $TC^* = TC$. Determine the vector of time multiples corresponding to the current value of F_{UB} so that:

$m_i^{UB}\left(m_i^{UB} - 1\right) \leq \left(T_i^*/F_{UB}\right)^2 \leq m_i^{UB}\left(m_i^{UB} + 1\right)$ for all i,

and compute the corresponding total cost, say TC_{UB}. If $TC_{UB} \leq TC^*$, then update the best solution found so far, update F_{UB} to $TC^* \big/ \sum_{i=1}^{n} h_i d_i m_i^{UB}$ and repeat this step; else, goto the next step.

Step 2: Initialize the lower bound on the basic period F, say F_{LB}, to $\max\left\{\min\limits_i \sqrt{a_i/h_i d_i}, 2s/TC^*\right\}$. Determine the vector of time multiples corresponding to the current value of F_{LB} so that:

$m_i^{LB}\left(m_i^{LB} - 1\right) \leq \left(T_i^*/F_{LB}\right)^2 \leq m_i^{LB}\left(m_i^{LB} + 1\right)$ for all i,

and compute the corresponding total cost, say TC_{LB}. If $TC_{LB} \leq TC^*$, then update the best solution found so far, update F_{LB} to $\max\left\{\min\limits_i \sqrt{a_i/h_i d_i}, 2\left(A + \sum_{i=1}^{n} \frac{a_i}{m_i^{LB}}\right) \Big/ TC^*\right\}$, and repeat this step; else, goto the next step.

Step 3: For each item i ($i = 1, \ldots, n$), compute the value of the basic period at which the value of the time multiple m_i^{UB} increases by 1; that is:

$F_i = T_i^* \Big/ \sqrt{m_i^{UB}\left(m_i^{UB} + 1\right)}$, where $T_i^* = \sqrt{2a_i/h_i d_i}$.

Step 4: Set the basic period $F = \max_i F_i$. If $F \leq F_{LB}$, then stop; else, goto Step 5.

Step 5: Let p denotes the item for which m_i^{UB} changes to $m_i^{UB} + 1$ at time F. Increase m_p^{UB} by 1 and update F_p accordingly. Compute the total cost TC corresponding to the modified vector of time multiples. If $TC \leq TC^*$, then update the best solution found so far accordingly, update F_{LB} to $\max\left\{\min\limits_i \sqrt{a_i/h_i d_i}, 2\left(A + \sum_{i=1}^{n} \frac{a_i}{m_i^{LB}}\right) \Big/ TC^*\right\}$, and goto Step 4; else, stop.

Appendix B

The Probability Distributions and Their Characteristics

1. Probability Distribution – Poisson

A counting process of independent successes (e.g., orders, purchases, calls) occurring at a rate l, where l represents the average number of successes per unit of time l is constant across orders; that is, the population is homogeneous with respect to the success rate.

Variate: Number of successes (e.g., orders) that occur in an interval

Pmf: $p_\lambda(x) = P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$; $x = 0, 1, ...$

Cdf: $P_\lambda(x) = P(X \leq x) = \sum_{n=0}^{x} \frac{e^{-\lambda}\lambda^x}{x!}$

Complementary cdf (Survival Function):

$$\bar{P}_\lambda(x) = 1 - P(X < x) = 1 - \sum_{n=0}^{x-1} \frac{e^{-\lambda}\lambda^n}{n!} = \sum_{n=x}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!}$$

Mean: $E(x) = \lambda$

Variance: $\sigma^2 = \lambda$

Typical applications: Modelling demand for an item over a fixed interval of time.

Notes: The Poisson distribution does not take explicit account of the probability of success p in that ONLY successes are counted. A Poisson variable may be used to approximate a binomial variable when n is large and p is small. In practice, the Poisson assumption has been validated for frequently purchased consumer goods.

Mathematical Relations:

These some properties of Poisson distribution have been taken from the book of Hardley and Whitin (1963).

$$xp_\lambda(x) = x\frac{\lambda^x}{x!}e^{-\lambda} = \lambda\frac{\lambda^{x-1}}{(x-1)!}e^{-\lambda} = \lambda p_\lambda(x-1) \tag{B.1}$$

$$\sum_{x=y}^{\infty} xp_\lambda(x) = \lambda\sum_{x=y}^{\infty} p_\lambda(x-1) \tag{B.2}$$

$$\sum_{x=0}^{y}(y-x)p_\lambda(x) = y - \lambda + \lambda\sum_{x=y}^{\infty} p_\lambda(x) - y\sum_{x=y+1}^{\infty} p_\lambda(x) \tag{B.3}$$

$$\sum_{x=y}^{\infty}(x-y)p_\lambda(x) = \lambda\sum_{x=y-1}^{\infty} p_\lambda(x) - y\sum_{x=y}^{\infty} p_\lambda(x) \tag{B.4}$$

$$\int_0^T p_{\lambda t}(y)dt = \frac{1}{\lambda}\sum_{x=y+1}^{\infty} p_{\lambda T}(x) \tag{B.5}$$

$$\int_0^T t^n p_{\lambda t}(y)dt = \frac{1}{\lambda^{n+1}} \frac{(n+y)!}{y!} \sum_{x=n+y+1}^{\infty} p_{\lambda T}(x), n = 0, 1, 2, ... \tag{B.6}$$

$$\begin{aligned} \int_0^T \bar{P}_{\lambda t}(y)dt &= \frac{1}{\lambda} \sum_{x=y+1}^{\infty} \bar{P}_{\lambda T}(x) \\ &= T\bar{P}_{\lambda T}(y) - \frac{y}{\lambda} \bar{P}_{\lambda T}(y+1) \end{aligned} \tag{B.7}$$

$$\begin{aligned} \int_0^T t^n \bar{P}_{\lambda t}(y)dt = \frac{1}{\lambda^{n+1}} \sum_{x=y}^{\infty} \frac{(n+x)!}{x!} \bar{P}_{\lambda T}(n+x+1) = \frac{T^{n+1}}{n+1} \bar{P}_{\lambda T}(y) \\ - \frac{1}{\lambda^{n+1}} \left(\frac{1}{n+1}\right) \frac{(n+y)!}{(y-1)!} \bar{P}_{\lambda T}(n+y+1) \end{aligned} \tag{B.8}$$

2. Probability Distribution Binomial

An experiment consisting of a discrete sequence of independent trials, where each trial (e.g., customer arrival, customer order, output of a production process) results in success (e.g., purchase, conform to quality standards) or failure (e.g., non-purchase, non-conform to quality standards or defect), where the probability of success (respectively, failure) for each trial, say p (respectively, $1-p$), is constant across the experiment; that is, the population is homogeneous with respect to probability of success p.

Variate: Number of successes in n independent Bernoulli trials.

Pmf: $p(x|n) = P(X = x) = \binom{n}{x}p^x(1-p)^{n-x}$, where $\binom{n}{x} = \frac{n!}{x!(n-x)!}$, and $x = 0, 1, ..., n$

Cdf: $P(x|n) = P(X \leq x) = \sum_{k=0}^{x}\binom{n}{k}p^k(1-p)^{n-k}$,

Complementary cdf (Survival Function):

$$\bar{P}(x|n) = 1 - P(X < x) = 1 - \sum_{k=0}^{x-1}\binom{n}{k}p^k(1-p)^{n-k},$$

Mean: $E(x) = np$

Variance: $\sigma^2 = np(1-p)$

Typical applications: Modelling the number of defects in n units produced

Notes: We denote a binomial distribution with n trials and success probability p by binomial(n; p). This distribution is right-skewed when $p < 0.5$, and left-skewed when $p > 0.5$ and symmetric when $p = 0.5$.

3. Expectation Value

Let N be a numerically-valued discrete random variable with probability distribution $P(N)$. The expected value $\mathbb{E}(N)$ is defined by

$$
\begin{aligned}
\mathbb{E}(N) &= \sum_{n=1}^{\infty} nP(N = n), \\
&= \sum_{n=1}^{\infty} P(N \geq n), \\
&= \sum_{n=0}^{\infty} P(N > n).
\end{aligned}
$$

Bibliography

[1] V. Aggarwal, *Grouping multi-item inventory using common cycle periods*, European Journal of Operational Research **17** (1984), 369–372.

[2] Y. Aksoy and S. S. Erenguc, *Multi-item inventory models with co-ordinated replenishments: A survey*, International Journal of Operations & Production Management **8** (1988), no. 1, 63–73.

[3] F. M. Andres and H. Emmons, *On the optimal packaging frequency of jointly replenished items*, Management Science **22** (1976), 1165–1166.

[4] S. Anily, *Multi-item replenishment and storage problem (mirsp): heuristics and bounds*, Operations Research **39** (1991), 233–243.

[5] E. Arkin, D. Joneja, and R. Roundy, *Computational complexity of uncapacitated multi-echelon production planning problems*, Operations Research Letters **8** (1989).

[6] D. Atkins and P. O. Iyogun, *Periodic versus can-order policies for coordinated multi-item inventory systems*, Management Science **34** (1988), 791–796.

[7] D. Atkins, M. Queyranne, and D. Sun, *98% effective power-of-two lot size policies for finite production rate assembly systems*, Operations Research **40** (1992), no. 6, 126–141.

[8] J. L. Balintfy, *On a basic class of multi-item inventory problems*, Management Science **10** (1964), 287–297.

[9] M. Bastian, *Joint replenishment in multi-item inventory systems*, Journal of the Operational Research Society **37** (1986), 1113–1120.

[10] Z. P. Bayindir, S. I. Birbil, and J. B. G. Frenk, *The joint replenishment problem with variable production costs*, European Journal of Operational Research **175** (2006).

[11] M. Ben-Daya and M. Hariga, *Comparative study of heuristics for the joint replenishment problem*, Omega **23** (1995), no. 3, 341–344.

[12] R. G. Brown, *Decision rules for inventory management*, Holt Reinhart and Winston, New York, 1967.

[13] B. C. Cha and I. K. Moon, *The joint replenishment problem with quantity discounts under constant demand*, OR Spectrum **27** (2005).

[14] A. K. Chakravarty, *Multi-item inventory aggregation into groups*, Journal of the Operational Research Society (1981), 19–26.

[15] ______, *Multi-item inventory aggregation into groups*, Journal of the Operational Research Society **32** (1981), 19–26.

[16] ______, *Joint inventory replenishments with group discounts based on invoice value*, Management Science **30** (1984), no. 9, 1105–1112.

[17] A. K. Chakravarty, J. B. Orlin, and U. G. Rothblum, *A partitioning problem with additive objective with an application to optimal inventory groupings for joint replenishment*, Operations Research **30** (1982), no. 5, 1018–1022.

[18] ______, *Consecutive optimizers for a partitioning problem with applications to optimal inventory groupings for joint replenishment*, Operations Research **33** (1985), 820–834.

[19] C. K. Chan, B. K-S. Cheung, and A. Langevin, *Solving the multi-buyer joint replenishment problem with a modified genetic algorithm*, Transportation Research Part B: Methodological **37** (2003), no. 3, 291–299.

[20] C. K. Chan, Y-O Li, C. To Ng, K-S Cheung, and A. Langevin, *Scheduling of multi-buyer joint replenishments*, International Journal of Production Economics **102** (2006), no. 1, 132–142.

[21] C. H. Chern, *A multi-product joint ordering model with dependent set-up cost*, Management Science **20** (1974), 1081–1091.

[22] J. S. Dagpunar, *Formulation of a multi-item single supplier inventory problem*, Journal of the Operational Research Society **33** (1982).

[23] A. Eynan and D. H. Kropp, *Periodic review and joint replenishmentin stochastic demand environments*, IIE Transactions **30** (1998).

[24] A. Federgruen, H. Groenevelt, and H. C. Tijms, *Coordinated replenishments in a multi-item inventory system with compound poisson demand*, Management Science **30** (1984), 344–357.

[25] A. Federgruen and M. Tzur, *The joint replenishment problem with time-varying costs and demands: Efficient asymptotic and e-optimal solutions*, Operations Research **42** (1994), 1067–1086.

[26] A. Federgruen and Y. S. Zheng, *The joint replenishment problem with general joint cost structures*, Operations Research **40** (1992), 384–403.

[27] J. B. G. Frenk, M. J. Kleijn, and R. Dekker, *An efficient algorithm for a generalized joint replenishment problem*, European Journal of Operational Research **118** (1999), 413–428.

[28] R. Fung and X. Ma, *A new method for joint replenishment problems*, Journal of the Operational Research Society **52** (2001).

[29] T. E. Goswick and B. D. Sivazlian, *The mixed ordering policy in periodic review stochastic multi-commodity inventory systems*, Naval Research Logistics Quarterly **21** (1974), 389–410.

[30] S. K. Goyal, *Economic packaging frequency for items jointly replenished*, Operations Research **21** (1973), no. 2, 644–647.

[31] ———, *A method for improving joint replenishment with a known frequency of replenishment orders*, International Journal of Production Research **11** (1973), 195–200.

[32] ———, *Determination of optimal packaging frequency of jointly replenished items*, Management Science **21** (1974), 436–443.

[33] ———, *Optimum ordering policy for a multi-item single supplier system*, Operational Research Quarterly **25** (1974), 293–298.

[34] ———, *Two methods for the analysis of a joint replenishment system with known order frequencies*, International Journal of Production Research **12** (1974), 721–724.

[35] ———, *Analysis of joint replenishment inventory systems with resource restriction*, Operational Research Quarterly **26** (1975), 197–203.

[36] ———, *Economic packaging frequency for finite horizon inventory problems*, International Journal of Production Research **13** (1975), 383–389.

[37] ———, *Economic packaging frequency of jointly replenished items having interdependent packaging set-up cost*, Operational Research Quarterly **27** (1976), 111–118.

[38] ———, *Economic packaging frequency of perishable jointly replenished*, Operational Research Quarterly **28** (1977), 215–219.

[39] ______, *Determination of economic packaging frequency in a multi-brand joint replenishment inventory system*, European Journal of Operational Research **4** (1980), 185–188.

[40] ______, *On improving silvers algorithm for the joint replenishment problem*, IIE Transactions **17** (1985), 99–101.

[41] ______, *A simple algorithm for determining the optimum policy for jointly replenished items*, Engineering Costs and Production Economics **11** (1987), 147–150.

[42] ______, *Determining the optimum production-packaging policy for jointly replenished items*, Engineering Costs and Production Economics **15** (1988), 339–341.

[43] ______, *Economic ordering policy for jointly replenished items*, International Journal of Production Research **26** (1988), 1237–1240.

[44] ______, *Determination of economic packaging frequency of items jointly replenished*, Management Science (20), 232–235.

[45] ______, *A comment on 'a new method for joint replenishment problems' by fung and ma*, The Journal of the Operational Research Society **53** (2002), no. 5, 592.

[46] S. K. Goyal and A. S. Belton, *On a simple method of determining order quantities in joint replenishments for deterministic demand*, Management Science **25** (1979), 604.

[47] S. K. Goyal and S. G. Deshmukh, *A note on 'the economic ordering quantity for jointly replenished items'*, International Journal of Production Research **31** (1993), 2959–2961.

[48] S. K. Goyal and A. T. Satir, *Joint replenishment inventory control: deterministic and stochastic models*, European Journal Operations Research **38** (1989), 2–13.

[49] S. K. Goyal and R. Soni, *Economic packaging frequency of jointly replenished items with multiple manufacturing and packaging cycles*, Proceedings of the third international symposium on inventories, 1984, pp. 521–529.

[50] S. C. Graves, *On the deterministic multi-product single-machine lot scheduling problem*, Management Science **25** (1979), 276–280.

[51] ______, *A multi-echelon inventory model for a repairable item with one-for-one replenishment*, Management Science (1985), 1247–1256.

[52] S. C. Graves and L. B. Schwarz, *Single cycle continuous review policies for arborescent production/inventory systems*, Management Science **23** (1977), 529–540.

[53] G. Hadley and T. M. Whitin, *Analysis of invenotry systems*, Prentice-Hall, Englewood Cliffs, NJ, 1963.

[54] D. J. Harfiel and G. L. Curry, *An algorithm for optimal inventory policies for systems with joint set-up costs*, Management Science (1974), 1175–1177.

[55] M. Hariga, *Two new heuristics procedures for the joint replenishment problem*, Journal of the Operational Research Society (1994), 463–471.

[56] S-P Hong and Y-H. Kim, *A genetic algorithm for joint replenishment based on the exact inventory cost*, Computers & Operations Research **36** (2009), 67175.

[57] M. A. Hoque, *An optimal solution technique for the joint replenishment problem with storage and transport capacities and budget constraints*, European Journal of Operational Research **175** (2006), 10331042.

[58] E. Ignall, *Optimal continuous review policies for two product inventory systems with joint set-up costs*, Management Science **15** (1969), 278–283.

[59] P. Jackson, W. Maxwell, and J. Muckstadt, *The joint replenishment problem with a powers-of-two restriction*, IIE Transactions **17** (1985), 25–32.

[60] R. Jayaraman and M. T. Tabucanon, *Coordinated versus independent replenishment inventory control*, International Journal of Operations and Production Management **4** (1984), 61–68.

[61] S. G. Johansen and P. Melchiors, *Can-order policy for the periodic-review joint replenishment problem*, Journal of the Operational Research Society **54** (2003), 283–290.

[62] E. L. Johnson, *Optimality and computation of (σ, S) policies in the multi-item infinite horizon inventory problem*, Management Science **13** (1967), no. 7, 475–491.

[63] M. Kaspi, *Determination of economic packaging set-up frequency of jointly replenished items*, The Journal of the Operational Research Society **42** (1991), no. 2, 165–168.

[64] M. Kaspi and M. J. Rosenblatt, *An improvement of silvers algorithm for the joint replenishment problem*, IIE Transactions **15** (1983), 264–267.

[65] ———, *The effectiveness of heuristic algorithms for multi-item inventory systems with joint replenishment costs*, International Journal of Production Research **23** (1985), 109–116.

[66] ———, *On the economic ordering quantity for jointly replenished items*, International Journal of Production Research **29** (1991), 107–114.

[67] M. Khouja and S. K. Goyal, *A review of the joint replenishment problem literature: 1989-2005*, European Journal of Operational Research **186** (2008), 1–16.

[68] M. Khouja, Z. Michalewicz, and S. S. Satoskar, *A comparison between genetic algorithms and the rand method for solving the joint replenishment problem*, 11 **6** (2006), 556–564.

[69] M. Khouja, S. Park, and C. Saydam, *Joint replenishment problem under continuous unit cost change*, International Journal of Production Research **43** (2005), 311–326.

[70] C. K. Klein and J. A. Ventura, *An optimal method for a deterministic joint replenishment inventory policy in discrete time*, The Journal of the Operational Research Society **46** (1995), no. 5, 649–657.

[71] C. Larsen, *Comments to Özkaya et al.(2006)*, University of Aarhus, Aarhus School of Business, Department of Business Studies, 2008.

[72] F. C. Lee and M. J. Yao, *A global optimum search algorithm for the joint replenishment problem under power-of-two policy*, Computers & Operations Research **30** (2003), 1319–1333.

[73] L. H. Lee and E. P. Chew, *A dynamic joint replenishment policy with auto-correlated demand*, European Journal of Operational Research **165** (2005), no. 3, 729–747.

[74] Q. Li, *Solving the multi-buyer joint replenishment problem with the rand method*, Computers & Industrial Engineering **46** (2004), no. 4, 755–762.

[75] R. A. Low and J. F. Waddington, *The determination of optimum joint replenishment policy for a group of discount-connected stock lines*, Operational Research Quarterly **18** (1977), no. 4, 443–462.

[76] P. Melchiors, *Calculating can-order policies for the joint replenishment problem by the compensation approach*, European Journal of Operational Research **141** (2002), 587–595.

[77] S. Minner and E. A. Silver, *Multi-product batch replenishment strategies under stochastic demand and a joint capacity constraint*, IIE Transactions **37** (2005), no. 5, 469–479.

[78] I. K. Moon and B. C. Cha, *The joint replenishment problem with resource restriction*, European Journal of Operational Research **173** (2006), no. 1, 190–198.

[79] E. Naddor, *Optimal and heuristic decisions in single and multi-item inventory systems*, Management Science **21** (1975), 1234–1249.

[80] C. Nielsen and C. Larsen, *An analytical study of the Q(s, S) policy applied to the joint replenishment problem*, European Journal of Operational Research **163** (2005), 721–732.

[81] D. J. Nocturne, *Economic ordering frequency for several items jointly replenished*, Management Science **19** (1973), 1093–1096.

[82] A. Olsen, *An evolutionary algorithm for the joint replenishment of inventory with interdependent ordering costs*, Genetic and evolutionary computationgecco 2003, 2003, pp. 2416–2417.

[83] A. L. Olsen, *An evolutionary algorithm to solve the joint replenishment problem using direct grouping*, Computers and industrial Engineering **48** (2005), 223–235.

[84] B. Y.l Özkaya, Ü. Gürler, and E. Berk, *The stochastic joint replenishment problem: a new policy, analysis, and insights*, Naval Research Logistics (NRL) **53** (2006), no. 6, 525–546.

[85] E. Page and R. J. Paul, *Multi-product inventory situations with one restriction*, Operational Research Quarterly **27** (1976), no. 4, 815–834.

[86] P. Pantumsinchai, *A comparison of three joint ordering inventory policies**, Decision Sciences **23** (1992), no. 1, 111–127.

[87] M. Queyranne, *Comment on a dynamic programming algorithm for joint replenishment under general order cost functions*, Management Science **33** (1987), no. 1, 131–133.

[88] M. Queyranne and D. Sun, *The performance ratio of grouping policies for the joint replenishment problem*, Discrete Applied Mathematics **46** (1993), 43–72.

[89] M. J. Rosenblatt and M. Kaspi, *A dynamic programming algorithm for joint replenishment under general order cost functions*, Management Science **3** (1985), no. 1, 369–373.

[90] S. M. Ross, *Applied probability models with optimization applications*, Courier Corporation, 1970.

[91] J-P Schaack and E. A. Silver, *A procedure, involving simulation, for selecting the control variables of an* (S, c, s) *joint ordering strategy*, Information Systems and Operational Research **10** (1972), 154–170.

[92] H. Schultz and S. G. Johansen, *Can-order polices for coordinated inventory replenishment with erlang distributed times between ordering*, European Journal of Operational Research (1999), 30–41.

[93] L. B. Schwarz, *A simple continuous review deterministic one-warehouse n-retailer inventory problem*, Management Science (1973), 555–566.

[94] l. B. Schwarz, M. Queyranne, and S. K. Goyal, *Comment on 'A dynamic programming algorithm for joint replenishment under general order cost functions'*, Management Science **33** (1987), 151–153.

[95] F. Shu, *Economic ordering frequency for two items jointly replenished*, Management Science **17** (1971), 406–410.

[96] H. Siajadi, R. N. Ibrahim, P. B. Lochert, and W. M. Chan, *Joint replenishment policy in inventory-production systems*, Production Planning and Control **16** (2005).

[97] E. A. Silver, *Some characteristics of a special joint-order inventory model*, Operations Research **14** (1965), 319–322.

[98] ______, *A control system for co-ordinated inventory replenishment*, International Journal of Production Research **12** (1974), 647–671.

[99] ______, *A simple method of determining order quantities in joint replenishments under deterministic demand*, Management Science **22** (1976), 1351–1361.

[100] ______, *Establishing reorder points in the (S, c, s) co-ordinated control system under compound poisson demand*, International Journal of Production Research **19** (1981), 743–750.

[101] D. M. Simmons, *Optimal inventory policies under a hierarchy of setup costs*, Management Science **18** (1972), 591–599.

[102] R. M. Thompstone and E. A. Silver, *A co-ordinated inventory control system for compound poisson demand and zero lead time*, International Journal of Production Research **13** (1975), 581–602.

[103] M. J. G. Van Eijs, *A note on the joint replenishment problem under constant demand*, Journal of the Operational Research Society **44** (1993), 185–191.

[104] M. J. G. Van Eijs, R. M. J. Heuts, and J. P. C. Kleijnen, *Analysis and comparison of two strategies for multi-item inventory systems with joint replenishment costs*, European Journal of Operations Research **59** (1992), 405–412.

[105] S. Viswanathan, *A new optimal algorithm for the joint replenishment problem*, Journal of the Operational Research Society **47** (1996), 936–44.

[106] ______, *Periodic review (s, S) policies for joint replenishment inventory systems*, Management Science **43** (1997), 1447–1454.

[107] ______, *On optimal algorithms for the joint replenishment problem*, Journal of the Operational Research Society **53** (2002), 1286–1290.

[108] R. E. Wildeman, J. B. G. Frenk, and R. Dekker, *An efficient optimal solution method for the joint replenished problem*, European Journal of Operational Research **99** (1997), 433–444.

[109] H. Xu, S-L. Jin, and Y. Hong, *An algorithm for the joint replenishment problem with multiple resource constraints*, Wireless communications, networking and mobile computing, 2007. wicom 2007. international conference on, 2007, pp. 6394–6397.

[110] Y. S. Zheng and A. Federgruen, *Finding optimal (s, S) policies is as simple as evaluating a single policy*, Operations Research **39** (1991), 654–665.

[111] P. H. Zipkin, *Foundations of inventory management*, Irwin/McGraw-Hill series: Operations and decision sciences, McGraw-Hill, 2000.